COMBINATORIAL MODELS IN THE TOPOLOGICAL CLASSIFICATION OF SINGULARITIES OF MAPPINGS

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Abstract. The topological classification of finitely determined map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is discrete (by a theorem due to R. Thom), hence we want to obtain combinatorial models which codify all the topological information of the map germ f. According to Fukuda's work, the topology of such germs is determined by the link, which is obtained by taking the intersection of the image of f with a small enough sphere centered at the origin. If $f^{-1}(0) = \{0\}$, then the link is a topologically stable map $\gamma : S^{n-1} \to S^{p-1}$ (or stable if (n, p) are nice dimensions) and f is topologically equivalent to the cone of γ. When $f^{-1}(0) \neq \{0\}$, the situation is more complicated. The link is a topologically stable map $\gamma : N \to S^{p-1}$, where N is a manifold with boundary of dimension $n-1$. However, in this case, we have to consider a generalized version of the cone, so that f is again topologically equivalent to the cone of the link diagram. We analyze some particular cases in low dimensions, where the combinatorial models are provided by objects which are well known in Computational Geometry, for instance, the Gauss word or the Reeb graph.

Contents

1. Introduction

René Thom showed in [41] that the topological classification of finitely determined map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is discrete and hence, there are no moduli. The same assertion is not true if we consider the C^∞ classification by $\mathscr{A}$-equivalence (for instance, consider the 1-parameter family $f_t(x, y) =$

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$xy(x+y)(x-ty)$) or if we remove the finite determinacy assumption. In fact, Thom himself found a 1-parameter family of germs $f_t : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ with the property that any two distinct members of the family are not topologically equivalent (see [40]). Since the classification problem is discrete, a natural open question is to find a good combinatorial model which codifies the topological information of the map germ.

In [8], T. Fukuda proved that if the map germ is finitely determined and has isolated zeros (i.e., if $f^{-1}(0) = \{0\}$), then f has a cone structure on its link. The link is obtained by intersecting the image of f with a small enough sphere centered at the origin in $\mathbb{R}^p$. The main result is that the link turns out to be a mapping between spheres $\gamma : S^{n-1} \to S^{p-1}$ which is topologically stable (in fact, stable if (n,p) are nice dimensions in Mather's sense). Moreover, f is topologically equivalent to the cone of its link. Thus, the topological classification of germs can be deduced from the topological classification of topological stable mappings between spheres of one dimension less. We remark that the condition of isolated zeros is automatically satisfied when $n \leq p$. We review the proof of Fukuda's cone structure theorem for germs with isolated zeros in Section 4.

In a later paper [9], Fukuda also considered the case of non isolated zeros (i.e., $f^{-1}(0) \neq \{0\}$). The classification problem in this case is much more complicated. He showed the link is a mapping $\gamma : N \to S^{p-1}$ from a manifold with boundary N which is again topologically stable (or stable in nice dimensions). However, the germ f has not a cone structure on its link in the usual sense. We introduce in Section 7 the notion of generalized cone (following [5]) and also give an adapted version of the cone theorem for the case of non isolated zeros, by using this generalized version of the cone.

In low dimensions, the topological classification of finitely determined map germs has been widely developed by the author and other collaborators. We have studied the cases of map germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ in [22, 23, 24], map germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ in [31, 32, 34] and map germs $f : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ in [33, 35, 36]. In all these cases, the combinatorial model used for the topological classification is provided by the Gauss words.

More recently, we have also considered the case of map germs $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ in [2, 3] where we use the Reeb graph as a good combinatorial model for the singularity and the case of map germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ in [25], where we find a connection with Knot Theory. In [5], we also consider the topological classification of map germs with respect to the contact equivalence $\mathscr{K}$ instead of the right-left equivalence $\mathscr{A}$.

Gauss words and Reeb graphs are well known objects in Computational Geometry. We will explain in Sections 5, 6 how to construct these models as well as the main results for the case of map germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ and map germs $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ with isolated zeros, respectively.

In Sections 2 and 3 we review the basic concepts of stability and finite determinacy that we need for the course. There are no proofs for all the results in these sections, but we provide precise references which can be found basically in the celebrated six papers about stability by Mather [16, 17, 18, 19, 20, 21], the survey papers by Wall [42, 43] and the book by Gibson et al. [11].

2. Stability

Along the text, we use the following notation:

$$\begin{aligned}\mathscr{E}_n &= \text{ set of } C^\infty \text{ function germs } h:(\mathbb{R}^n,0)\to\mathbb{R},\\ \mathscr{E}(n,p) &= \text{ set of } C^\infty \text{ map germs } f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0),\\ \mathscr{R}_n &= \text{ set of } C^\infty \text{ diffeomorphism germs } \phi:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0).\end{aligned}$$

Definition 2.1. We say that two germs $f,g\in\mathscr{E}(n,p)$ are *$\mathscr{A}$-equivalent* if there exist $\phi\in\mathscr{R}_n$ and $\psi\in\mathscr{R}_p$ such that $g=\psi\circ f\circ\phi^{-1}$. That is, the following diagram is commutative, where the columns are diffeomorphisms:

$$\begin{array}{ccc}(\mathbb{R}^n,0) & \xrightarrow{f} & (\mathbb{R}^p,0)\\ \Big\downarrow\phi & & \Big\downarrow\psi\\ (\mathbb{R}^n,0) & \xrightarrow{g} & (\mathbb{R}^p,0)\end{array}$$

In the case that ϕ,ψ are homeomorphisms instead of diffeomorphisms, then we say that f,g are *C^0-$\mathscr{A}$-equivalent.*

We can characterize the $\mathscr{A}$-equivalence through the group action $\mathscr{R}_n\times\mathscr{R}_p$ on $\mathscr{E}(n,p)$ given by $(\phi,\psi)\cdot f=\psi\circ f\circ\phi^{-1}$. Then $f,g\in\mathscr{E}(n,p)$ are $\mathscr{A}$-equivalent if they are in the same orbit.

Given $f\in\mathscr{E}(n,p)$, an r-parameter *unfolding* of f is another germ $F\in\mathscr{E}(r+n,r+p)$ of the form $F(u;x)=(u;f_u(x))$ and such that $f_0=f$.

Definition 2.2. Two unfolding F,G of f are *$\mathscr{A}$-equivalent* (as unfoldings) if there exist diffeomorphisms $\Phi\in\mathscr{R}_{r+n}$ and $\Psi\in\mathscr{R}_{r+p}$ unfoldings of the identity maps in $(\mathbb{R}^n,0)$ and $(\mathbb{R}^p,0)$ respectively, such that $G=\Psi\circ F\circ\Phi^{-1}$.

We say that an unfolding F of f is *trivial* if F is $\mathscr{A}$-equivalent to the constant unfolding $G=\mathrm{id}\times f$. If $F(u;x)=(u;f_u(x))$, $\Phi(u;x)=(u;\phi_u(x))$ and $\Psi(u;y)=(u;\psi_u(y))$, we have

$$\psi_u\circ f_u\circ\phi_u^{-1}=f.$$

Thus, the germ of f_u at the point $\phi_u^{-1}(0)$ is $\mathscr{A}$-equivalent to f, but in general we do not have $\phi_u^{-1}(0)=0$.

We say that $f\in\mathscr{E}(n,p)$ is *stable* if any unfolding F of f is trivial.

The above definition is known as stability by deformations or by homotopies when we consider mappings instead of germs. An immediate consequence of the definition is that the property that a germ is stable is invariant under $\mathscr{A}$-equivalence. Therefore, the definition can be extended without problem by taking coordinate charts for smooth map germs between smooth manifolds $f:(N,x)\to(P,y)$.

Exemple 2.3. From the definition, one deduces easily that if f is regular (i.e., the differential df_0 has maximal rank), then f is stable.

The set of germs $\mathscr{E}_n$ has a structure of commutative and unit local $\mathbb{R}$-algebra, whose maximal ideal $\mathfrak{m}_n$ is given by the germs $h\in\mathscr{E}_n$ such that $h(0)=0$. Any $f\in\mathscr{E}(n,p)$ induces an $\mathbb{R}$-algebra homomorphism $f^*:\mathscr{E}_p\to\mathscr{E}_n$ through $f^*(h)=h\circ f$. Moreover, $\mathscr{E}(n,p)$ has a structure of $\mathscr{E}_n$-module and of $\mathscr{E}_p$-module via f^*.

Given $f \in \mathscr{E}(n,p)$, we denote by $\theta(f)$ the set of C^∞ germs of vector fields $\eta : (\mathbb{R}^n, 0) \to T\mathbb{R}^p$ along f, that is, such that $\pi \circ \eta = f$ where $\pi : T\mathbb{R}^p \to \mathbb{R}^p$ is the canonical projection. A generic element of $\theta(f)$ is written in a unique way as

$$\eta = \sum_{i=1}^{p} g_i \left(\frac{\partial}{\partial y_i} \circ f \right), \quad g_i \in \mathscr{E}_n,$$

where $y_1, \dots, y_p$ are the coordinates in $\mathbb{R}^p$. In this way, $\theta(f)$ has a structure of $\mathscr{E}_n$-module isomorphic to $(\mathscr{E}_n)^p$, after identification of η with the p-tuple $(g_1, \dots, g_p)$. In case that f is the germ of the identity map in $(\mathbb{R}^n, 0)$ or $(\mathbb{R}^p, 0)$, we denote $\theta(f)$ by θ_n or θ_p respectively.

For each $f \in \mathscr{E}(n,p)$ we can define two module homomorphisms. First, we have an $\mathscr{E}_n$-module homomorphism:

$$\begin{array}{rcl} tf : \theta_n & \to & \theta(f) \\ \xi & \mapsto & df \circ \xi \end{array},$$

where df is the differential of f. On the other hand, we have an $\mathscr{E}_p$-module homomorphism:

$$\begin{array}{rcl} wf : \theta_p & \to & \theta(f) \\ \eta & \mapsto & \eta \circ f \end{array},$$

where now in $\theta(f)$ we consider the $\mathscr{E}_p$-module structure induced by $f^* : \mathscr{E}_p \to \mathscr{E}_n$.

Definition 2.4. Given $f \in \mathscr{E}(n,p)$, the *$\mathscr{A}$-tangent space* (of f) and the *extended $\mathscr{A}$-tangent space* (of f) are defined respectively as

$$T\mathscr{A}f = tf\big(\mathfrak{m}_n\theta_n\big) + wf\big(\mathfrak{m}_p\theta_p\big),$$
$$T\mathscr{A}_ef = tf\big(\theta_n\big) + wf\big(\theta_p\big).$$

The *$\mathscr{A}$-codimension* and the *$\mathscr{A}_e$-codimension* are defined as

$$\mathscr{A} - \operatorname{codim}(f) = \dim_{\mathbb{R}} \frac{\mathfrak{m}_n\theta(f)}{T\mathscr{A}f},$$
$$\mathscr{A}_e - \operatorname{codim}(f) = \dim_{\mathbb{R}} \frac{\theta(f)}{T\mathscr{A}_ef}.$$

The following theorem is known as the infinitesimal stability criterion of Mather. The proof can be found in [11, 2.2].

Theorem 2.5. *A germ $f \in \mathscr{E}(n,p)$ is stable if and only if its $\mathscr{A}_e$-codimension is zero.*

By using the above identification of $\theta(f)$ with $(\mathscr{E}_n)^p$, the above theorem says that $f \in \mathscr{E}(n,p)$ is stable if and only if for each $\alpha \in (\mathscr{E}_n)^p$ there exist $g \in (\mathscr{E}_n)^n$ and $h \in (\mathscr{E}_p)^p$ such that

$$\alpha = h \circ f + \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}.$$

Exemple 2.6. We begin with the case of functions $p = 1$, we see that $f \in \mathscr{E}(n,1)$ has stable singularity if and only f is a Morse function. In fact, if f is a Morse function (i.e., it has non degenerate critical point at the origin) by the Morse lemma, we can assume that f is given (up to coordinate changes) by

$$f(x) = x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_n^2,$$

in such a way that $\frac{\partial f}{\partial x_i} = \pm 2x_i$. Given $\alpha \in \mathscr{E}_n$, by the Hadamard lemma there exist $g_i \in \mathscr{E}_n$ such that α is written as

$$\alpha = \alpha(0) + \sum_{i=1}^{n} x_i g_i = \alpha(0) \circ f + \sum_{i=1}^{n} (\pm \frac{g_i}{2}) \frac{\partial f}{\partial x_i},$$

and f is stable by Theorem 2.5.

Conversely, suppose that f is not a Morse function and has a denegerate critical point at the origin. Consider the n-parameter unfolding $F(a, x) = (a, f_a(x))$ given by

$$f_a(x) = f(x) + a_1 x_1 + \cdots + a_n x_n.$$

Fix a representative $F : \mathbb{R}^n \times U \to \mathbb{R}^n \times \mathbb{R}$, where $U \subset \mathbb{R}^n$ is an open neighbourhood of the origin. By the Thom Transversality Theorem [12, Theorem 4.9], for almost any $a \in \mathbb{R}^n$, $f_a : U \to \mathbb{R}$ is a Morse function, and hence, f_a cannot be $\mathscr{A}$-equivalent to f. This shows that the unfolding F is not trivial and f is not stable.

Exemple 2.7. Let $n = 1$ and $p = 2$. Then $f \in \mathscr{E}(1, 2)$ is stable only in the case that f is an immersion. In fact, if f is singular we can consider the 2-parameter unfolding $F(a, x) = (a, f_a(x))$ given by

$$f_a(x) = f(x) + ax.$$

Again we fix a representative $F : \mathbb{R}^2 \times (-\epsilon, \epsilon) \to \mathbb{R}^2 \times \mathbb{R}^2$. By the Thom Transversality Theorem, for almost any $a \in \mathbb{R}^2$, $f_a : (-\epsilon, \epsilon) \to \mathbb{R}^2$ is an immersion and f_a cannot be $\mathscr{A}$-equivalent to f. Thus, F is not trivial and f is not stable.

Sometimes it can be complicated to see that a certain germ is stable by means of Theorem 2.5. Also, the genericity arguments we have used to check that the germs are the only stable singularities may not work in higher dimensions. We present here a pair of results which give easy methods to check stability and to obtain normal forms for stable germs. Both are related to the concept of contact or $\mathscr{K}$-equivalence. This is another important equivalence introduced by Mather, which is weaker than $\mathscr{A}$-equivalence. We do not include here the definition, details can be found in [19].

Definition 2.8. For each germ $f \in \mathscr{E}(n, p)$, the *$\mathscr{K}$-extended tangent space* is defined as:

$$T\mathscr{K}_e f = tf(\theta_n) + (f^*\mathfrak{m}_p)\theta(f).$$

Note that $\omega f(\mathfrak{m}_p \theta_p) \subset (f^*\mathfrak{m}_p)\theta(f)$, hence ωf induces a well defined morphism:

$$\overline{\omega} f : \mathbb{R}^p \cong \frac{\theta_p}{\mathfrak{m}_p \theta_p} \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f}.$$

Lemma 2.9. [19, Proof of Proposition I.6] *A germ $f \in \mathscr{E}(n, p)$ is stable if and only if $\overline{\omega}(f)$ is an epimorphism.*

An equivalent statement of Lemma 2.9 is that f is stable if and only if $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the classes of the canonical basis $\{e_1,\dots,e_p\}$ in $\mathbb{R}^p$. Note that $T\mathscr{K}_e f$ is an $\mathscr{E}_n$-module which is finitely generated, in fact, it is generated over $\mathscr{E}_n$ by $\partial f/\partial x_i$, $i=1,\dots,n$ and by $f_j e_k$, with $j,k=1,\dots,p$. Thus, it is possible to compute it by using some computer algebra system like Singular [13].

Definition 2.10. For each germ $f\in\mathscr{E}(n,p)$, the *local algebra* (of f) is defined as
$$Q(f)=\frac{\mathscr{E}_n}{f^*\mathfrak{m}_p}.$$

Theorem 2.11. [19] *Two stable germs are $\mathscr{A}$-equivalent if and only if their local algebras are isomorphic.*

Exemple 2.12. Let us see that for $n=p=2$, a singular germ $f\in\mathscr{E}(2,2)$ is stable if and only if f has a singularity of type fold $f(x,y)=(x,y^2)$ or cusp $f(x,y)=(x,xy+y^3)$.

If f is a fold, we have:
$$\begin{aligned}T\mathscr{K}_e f&=\mathscr{E}_2\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\2y\end{pmatrix}\right\}+\langle x,y^2\rangle\mathscr{E}_2^2\\&=\mathscr{E}_2\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}0\\y\end{pmatrix}\right\}.\end{aligned}$$
Thus $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the class of $(0,1)$ and the map $\overline{\omega}f$ is obviously surjective, so f is stable by Lemma 2.9.

In the case of the cusp, we have:
$$\begin{aligned}T\mathscr{K}_e f&=\mathscr{E}_2\left\{\begin{pmatrix}1\\y\end{pmatrix},\begin{pmatrix}0\\3y^2+x\end{pmatrix}\right\}+\langle x,y^3+xy\rangle\mathscr{E}_2^2\\&=\mathscr{E}_3\left\{\begin{pmatrix}1\\y\end{pmatrix},\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}0\\y^2\end{pmatrix}\right\}.\end{aligned}$$
Now, $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the classes of $\{(1,0),(0,1)\}$. Again $\overline{\omega}f$ is surjective and hence, f is stable.

Assume now that f is stable, so that $\overline{\omega}f$ is surjective. If f has rank 0, then $T\mathscr{K}_e f\subset\mathfrak{m}_2\theta(f)$. Since $\theta(f)/\mathfrak{m}_2\theta(f)$ has dimension 2, we must have necessarily that $T\mathscr{K}_e f=\mathfrak{m}_2\theta(f)$. Moreover, $(f^*\mathfrak{m}_2)\subset\mathfrak{m}_2^2\theta(f)$, the classes of $\partial f/\partial x$ and $\partial f/\partial y$ should generate $\mathfrak{m}_2\theta(f)/\mathfrak{m}_2^2\theta(f)$ over $\mathbb{R}$. But this is not possible, since this space has dimension 4.

Thus, f must have rank 1 and after a coordinate change in the source, we can assume that $f(x,y)=(x,g(x,y))$, for some function $g\in\mathfrak{m}_2^2$. In other words, we see f as an unfolding of $g_0(y)=g(0,y)$. An easy exercise shows that
$$\frac{\theta(f)}{T\mathscr{K}_e(f)}\cong\frac{\theta(g_0)}{T\mathscr{K}_e(g_0)}\cong\frac{\mathscr{E}_1}{\langle g_0'\rangle}.$$
If $g_0\in\mathfrak{m}_1^4$, then $g_0'\in\mathfrak{m}_1^3$ and thus $\dim_{\mathbb{R}}(\mathscr{E}_1/\langle g_0'\rangle)\geq 3$, which is not possible by the surjectivity of $\overline{\omega}f$. Hence, g_0 must have order 2 or 3. But this implies that either $Q(f)\cong\mathscr{E}_1/\langle y^2\rangle$ or $Q(f)\cong\mathscr{E}_1/\langle y^3\rangle$. By Theorem 2.11, f is $\mathscr{A}$-equivalent either to the fold or the cusp, respectively.

Given a germ $f \in \mathscr{E}(n,p)$, for each $k \in \mathbb{N}$ we denote by $j^k f(0)$ the k-jet of f, that is, the Taylor polynomial of degree k of f at the origin. The k-jet space $J^k(n,p)$ is the space of k-jets $j^k f(0)$ of germs $f \in \mathscr{E}(n,p)$. Then $J^k(n,p)$ is identified with the space of polynomial maps $\sigma : \mathbb{R}^n \to \mathbb{R}^p$ of degree less than or equal to k and such that $\sigma(0) = 0$. We denote by $L^k(n) \subset J^k(n,n)$ the group of k-jets of diffeomorphism germs with the product defined by the k-jet of the composition. Moreover, we have the action of $L^k(n) \times L^k(p)$ on $J^k(n,p)$ induced by the action of $\mathscr{R}_n \times \mathscr{R}_p$ on $\mathscr{E}(n,p)$.

The k-jet spaces provide a finite-dimensional model of the classification problem by $\mathscr{A}$-equivalence. The jet space $J^k(n,p)$ can be identified with an Euclidean space $\mathbb{R}^N$ and the group $G = L^k(n) \times L^k(p)$ is a Lie group of finite dimension acting on $J^k(n,p)$ in a semialgebraic way. As a consequence, for each $\sigma \in J^k(n,p)$ the orbit $G \cdot \sigma$ is a semialgebraic submanifold of $J^k(n,p)$. In fact, $G \cdot \sigma$ is a semialgebraic subset which contains at least a regular point. But the orbit is locally diffeomorphic at all of its points because of the group action. Thus, $G \cdot \sigma$ is regular at all of its points.

For each $f \in \mathscr{E}(n,p)$, we have an epimorphism from $\mathfrak{m}_n\theta(f)$ to $J^k(n,p)$ given by $g \mapsto j^k g(0)$ and whose kernel is $\mathfrak{m}_n^{k+1}\theta(f)$. This allows us to identify

$$T_\sigma(J^k(n,p)) \cong \frac{\mathfrak{m}_n\theta(f)}{\mathfrak{m}_n^{k+1}\theta(f)}.$$

Under this identification, the tangent space to the orbit $G \cdot \sigma$ is precisely

$$T_\sigma(G \cdot \sigma) = \frac{T\mathscr{A}f + \mathfrak{m}_n^{k+1}\theta(f)}{\mathfrak{m}_n^{k+1}\theta(f)}.$$

In particular, if $\mathfrak{m}_n^{k+1}\theta(f) \subset T\mathscr{A}f$, we deduce that the codimension of the orbit $G \cdot \sigma$ is equal to the $\mathscr{A}$-codimension.

Given $f \in \mathscr{E}(n,p)$, we denote by $j^k f : (\mathbb{R}^n, 0) \to J^k(n,p)$ the germ of the k-jet extension of f (for each $x \in \mathbb{R}^n$, $j^k f(x)$ is the k-jet of f at the point x after translation to the origin). Then, we have the following result which characterizes the stability in terms of k-jets.

Theorem 2.13. *Let $f \in \mathscr{E}(n,p)$, $k \geq p+1$ and $\sigma = j^k f(0)$. Then:*

(1) *f is stable if and only if $j^k f : (\mathbb{R}^n, 0) \to J^k(n,p)$ is transverse to $G \cdot \sigma$.*
(2) *If f is stable, then $\mathfrak{m}_n^{k+1}\theta(f) \subset T\mathscr{A}f$ and hence,* $\operatorname{codim}(G \cdot \sigma) = \mathscr{A} - \operatorname{codim}(f)$.
(3) *If f is stable, then g is $\mathscr{A}$-equivalent to f if and only if $j^k g(0) \in G \cdot \sigma$.*

Proof. Part (1) can be found in [43, Theorem 15] whilst (2) and (3) follow from the fact that if f is stable then it is $(p+1)$-determined (see again [43, Theorem 15] and Section 3), then use [42, Theorem 1.2]. □

Definition 2.14. Given a stable germ $f \in \mathscr{E}(n,p)$, we denote by $A \subset (\mathbb{R}^n, 0)$ the germ of the subset of points x such that the germ of f at x is $\mathscr{A}$-equivalent to the germ of f at 0. As a consequence of Theorem 2.13, A is the germ of a submanifold in $(\mathbb{R}^n, 0)$ of codimension $\mathscr{A} - \operatorname{codim}(f)$ and the restriction $f|_A : A \to (\mathbb{R}^p, 0)$ is an immersion (unless the trivial case that f is a submersion). We say that A is the *analytic stratum of f in the source.*

Note that if f is defined by a polynomial map, then A is a semialgebraic subset of $\mathbb{R}^n$.

We pass now to the study of multi-germs. Given a finite subset $S = \{x_1, \dots, x_r\} \subset \mathbb{R}^n$, we consider multi-germs of C^∞ maps of the form $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$. The definitions of $\mathscr{A}$-equivalence, unfolding, trivial unfolding and stable germ can be generalized without any problem for multi-germs. Also the definitions of $\theta(f)$ and of extended $\mathscr{A}$-tangent space $T_e\mathscr{A}f$ can be adapted to the case of multi-germs and the infinitesimal stability criterion (Theorem 2.5) is still true. Moreover, next theorem allows to check in a relatively easy way whether a multi-germ is stable.

Given a multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ with $S = \{x_1, \dots, x_r\}$, we denote the restriction germ by $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, y)$ and by A_i the analytic stratum of f_i in the source, $i = 1, \dots, r$.

Theorem 2.15. [19, 1.6] *A multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ is stable if and only if for each $i = 1, \dots, r$, f_i is stable and the subspaces*

$$df_{x_1}(T_{x_1}A_1), \dots, df_{x_r}(T_{x_r}A_r)$$

have regular intersection in $\mathbb{R}^p$.

The regular intersection condition means that the codimension of the intersection is the sum of all the codimensions. Note that if f is a submersion at a point x_i, then $df_{x_i}(T_{x_i}A_i) = \mathbb{R}^p$. In this way, if $\tilde{S} \subset S$ is the subset of critical (i.e., non submersive) points of f, $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ is stable if and only if $f : (\mathbb{R}^n, \tilde{S}) \to (\mathbb{R}^p, y)$ is stable. Thus, we can assume without loss of generality that all the points of S are critical.

Exemple 2.16. Let us see what happens in the above examples when we consider multi-germs.

- Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}, y)$. At each critical point x_i of S, f must be a Morse singularity and the analytic stratum A_i is only the point $\{x_i\}$. Hence, f is stable if and only if S is a single point and f has a Morse singularity at that point.
- Let $f : (\mathbb{R}, S) \to (\mathbb{R}^2, y)$. Then f is stable only in the case that it is an immersion with normal crossings. So, the stable multi-germs are the simple regular point and the transverse double point.
- Let $f : (\mathbb{R}^2, S) \to (\mathbb{R}^2, y)$. At each critical point x_i of S, f must have fold or cusp type. If any of the points x_i has cusp type, then again the analytic stratum is $\{x_i\}$ and necessarily $S = \{x_i\}$. Otherwise, if all the points have fold type, then each A_i is a curve and now the regular intersection condition implies that we can have either simple points or transverse double points. In conclusion, f is stable if and only if S is made of a simple fold, a simple cusp or two transverse folds.

Given a C^∞-mapping $f : N \to P$ between manifolds, we denote by $\Sigma(f) \subset N$ the set of critical points (where f is not submersive) and its image $\Delta(f) = f(\Sigma(f))$ is called the discriminant.

Definition 2.17. Given a stable multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, we denote by $B \subset (\mathbb{R}^p, y)$ the germ of all points $y' \in \Delta(f)$ such that the multi-germ of f at $S' = f^{-1}(y') \cap \Sigma(f)$ is $\mathscr{A}$-equivalent to the multi-germ of f at S. Then B is a germ of submanifold in $(\mathbb{R}^p, y)$ which results from the intersection of the submanifolds $f_i(A_i)$, where each A_i is the analytic stratum of $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, y)$ in the source. We say that B is the *analytic stratum of f in the target.*

If $\dim B = d$, then we say that the multi-germ f represents a d-dimensional stable type. In particular, when $d = 0$, we say that f is a 0-stable type. Again, in the case that f is polynomial, B is a semialgebraic subset of $\mathbb{R}^p$.

Next, we will prove an interesting property which will be used in Section 4 to construct the cone structure of finitely determined germs.

Definition 2.18. For each germ $f \in \mathscr{E}(n, p)$, we define $\tau(f)$ as the subspace of $\mathbb{R}^p$ given by the kernel of $\overline{\omega} f : \mathbb{R}^p \to \theta(f)/T\mathscr{K}_e f$.

We will see that if f is stable, then $\tau(f)$ is nothing but $T_0 B$, where B is the analytic stratum of f in the target. The first step is to prove that they have the same dimension.

Lemma 2.19. *If $f \in \mathscr{E}(n, p)$ is stable, then $\dim B = \dim_{\mathbb{R}} \tau(f)$, where B is the analytic stratum in the target.*

Proof. Assume that $\dim_{\mathbb{R}} \tau(f) = d$. By Lemma 2.9, we have

$$\dim_{\mathbb{R}} \frac{\theta(f)}{T\mathscr{K}_e f} = \dim_{\mathbb{R}} \frac{\mathbb{R}^p}{\tau(f)} = p - d.$$

We use the formulas of [42, 4.5.1, 4.5.2], which give:

$$\mathscr{A} - \operatorname{codim}(f) = p - d + (n - p) = n - d.$$

It follows from Theorem 2.13 that this is equal to the codimension of the analytic straum in the source in $(\mathbb{R}^n, 0)$. Thus, the analyic stratum in the source (and hence in the target) has dimension d. $\square$

Lemma 2.20. *Let $f \in \mathscr{E}(n, p)$ be a stable germ and assume that $\dim_{\mathbb{R}} \tau(f) = d$. Then f is $\mathscr{A}$-equivalent to $\mathrm{id}_{\mathbb{R}^d,0} \times g_0$, where $g_0 \in \mathscr{E}(n - d, p - d)$ is also a stable germ.*

Proof. We choose linear coordinates in $\mathbb{R}^p$ such that $\tau(f) = \mathbb{R}^d \times \{0\}$. Given $v \in \tau(f)$, there exists $\xi \in \theta_p$ such that $\xi_0 = v$ and $\omega f(\xi) = tf(\eta) + \nu$, for some $\eta \in \theta_n$ and $\nu \in f^*\mathfrak{m}_p\theta(f)$ and evaluating at 0, we get $v = df(\eta_0)$. This shows that $\tau(f) \subset df_0(\mathbb{R}^n)$. Hence, we can choose smooth coordinates in $(\mathbb{R}^n, 0)$ such that f is an unfolding of a map germ $g \in \mathscr{E}(n - d, p - d)$, that is,

$$f(u, x) = (u, g_u(x)),\ u \in \mathbb{R}^d,\ x \in \mathbb{R}^{n-d}.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\frac{\mathbb{R}^p}{\tau(f)} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \frac{\theta(f)}{T\mathscr{K}_e(f)} \arrow[d, "\cong"] \\
\mathbb{R}^{p-d} \arrow[r, "\overline{\omega} g_0"] & \frac{\theta(g_0)}{T\mathscr{K}_e(g_0)}
\end{tikzcd}$$

The top arrow is an isomorphism induced by $\overline{\omega} f$ and the columns are also isomorphisms: the left arrow is induced by the projection and is an isomorphism because $\tau(f)=\mathbb{R}^d\times\{0\}$ and the right arrow is also an isomorphism because f is an unfolding of g_0. Then, $\overline{\omega} g_0$ is also an isomorphism, so g_0 is stable by Lemma 2.9. By definition of stability, f is $\mathscr{A}$-equivalent to the constant unfolding $\operatorname{id}_{\mathbb{R}^d,0}\times g_0$. □

Corollary 2.21. *If $f\in\mathscr{E}(n,p)$ is stable, then $\tau(f)=T_0B$, where B is the analytic stratum in the target.*

Proof. By Lema 2.20, we can assume that $f=\operatorname{id}_{\mathbb{R}^d,0}\times g_0$, where $g_0\in\mathscr{E}(n-d,p-d)$ is also a stable germ such that $\tau(g_0)=\{0\}$. We know from Lemma 2.19 that the analytic stratum of g_0 is also equal to $\{0\}$, so

$$\tau(f)=\mathbb{R}^d\times\{0\}=T_0B.$$

□

Proposition 2.22. *Let $f:(\mathbb{R}^n,S)\to(\mathbb{R}^p,y)$ be a stable multi-germ and let B be the analytic stratum in the target. If $P\subset\mathbb{R}^p$ is a submanifold transverse to B at y, then $N=f^{-1}(P)$ is a submanifold of $\mathbb{R}^n$ in a neighbourhood of S and the restriction $f|_{N,S}:(N,S)\to(P,y)$ is stable.*

Proof. For each i, we have that $df_{x_i}(\mathbb{R}^n)\supset df_{x_i}(T_{x_i}A_i)\supset T_yB$. If B and P are transverse at y, then f is transverse to P at x_i and N is a submanifold of $\mathbb{R}^n$ in a neighbourhood of x_i. Let us assume for a moment that each restriction $f|_{N,x_i}:(N,x_i)\to(P,y)$ is a stable germ. The analytic stratum in the source is $N\cap A_i$ and the image by the differential of the tangent space is

$$df_{x_i}(T_{x_i}(N\cap A_i))=df_{x_i}(T_{x_i}A_i)\cap T_yP.$$

Thus, the transversality between B and P at the point y ensures that the images of the tangent spaces of $N\cap A_i$ have regular intersection in T_yP.

It only remains to show that each germ $f|_{N,x_i}:(N,x_i)\to(P,y)$ is stable. We assume, for simplicity, that $x_i=0$ and $y=0$. By Lemma 2.20, we can also assume that $f_i=\operatorname{id}_{\mathbb{R}^d,0}\times g$, where $g\in\mathscr{E}(n-d,p-d)$ is also a stable germ and $d=\dim A_i$. The transversality assumption implies that T_0P contains $\{0\}\times\mathbb{R}^{p-d}$ and T_0N contains $\{0\}\times\mathbb{R}^{n-d}$.

Consider the following diagram:

$$\begin{array}{ccc}(N,0) & \xrightarrow{f|_{N,0}} & (P,0)\\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j}\\ (\mathbb{R}^{n-d},0) & \xrightarrow{g} & (\mathbb{R}^{p-d},0),\end{array}$$

where $i(z)=(0,z)$ and $j(w)=(0,w)$. Then we have that i,j are immersions, that j is transverse to $f|_{N,0}$ and that diagram is cartesian (that is, it is commutative and the mapping (i,g) from $\mathbb{R}^{n-d}$ into the submanifold $\{(x,w)\in N\times\mathbb{R}^{p-d}:\ f(x)=j(w)\}$ is a diffeomorphism). Thus, $f|_{N,0}$ can be seen, after $\mathscr{A}$-equivalence, as an unfolding of g (see [11, III.0.1]). But it is easy to see that if g is stable, then any unfolding of g is also stable. □

If instead of germs of C^∞ maps, we consider germs of analytic maps (real or complex), then all the definitions of $\mathscr{A}$-equivalence, unfoldings, stability, extended $\mathscr{A}$-tangent space, $\mathscr{A}_e$-codimension as well as all the theorems relating these concepts are still valid. In that case, the diffeomorphisms, vector fields and manifolds are considered also of analytic class (real or complex). Moreover, all the showed examples of stable germs or multi-germs work in the same way in the analytic case (real or complex). In fact, we have the following result which gives the relation between the three classes [42, 1.7].

Proposition 2.23. *Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ be a real analytic multi-germ, then the following statements are equivalent:*

(1) *f is stable as a C^∞ multi-germ.*
(2) *f is stable as a real analytic multi-germ.*
(3) *The complexification $\hat{f}$ is stable as a complex analytic multi-germ.*

We finish this section with the notion of stability of mappings.

Definition 2.24. We say that $f : N \to P$ is *locally stable* if

(1) the restriction $f : \Sigma(f) \to P$ is finite (i.e., finite-to-one and closed),
(2) for any $y \in \Delta(f)$, the multi-germ $f : (N, S) \to (P, y)$ is stable, where $S = f^{-1}(y) \cap \Sigma(f)$.

Exemple 2.25. We come back to the above examples. Let $f : N \to P$, with $\dim N = n$ and $\dim P = p$.

- If $p = 1$, then f is locally stable if and only if f is a Morse function with distinct critical points (see fig. 1).

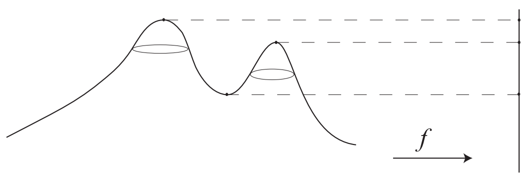


FIGURE 1. Example of locally stable function

- If $n = 1$ and $p = 2$, then f is locally stable if and only if f is an immersion with transverse double points (see fig. 2).

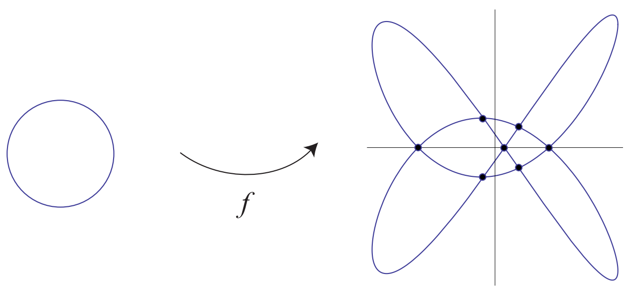


FIGURE 2. Example of locally stable plane curve

- If $n = p = 2$, then f is locally stable if and only if the singularities of f are simple folds, simple cusps of transverse double folds (see fig. 3).

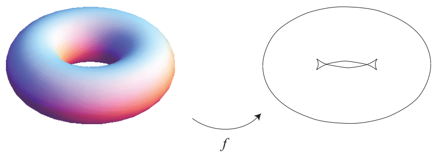


FIGURE 3. Example of locally stable map between surfaces

There exists a concept of global stability. We say that a C^∞ mapping $f : N \to P$ between smooth manifolds is *globally stable* if there exists a neighbourhood W of f in $C^\infty(N, P)$ with the Whitney C^∞ topology, such that any $g \in W$ is $\mathscr{A}$-equivalent to f. Mather proved in [17] that if the restriction $f|_{\Sigma(f)}$ is proper, then the local and the global stability coincide. However, this result cannot be used in the real or complex analytic case.

3. Finite determinacy

We begin this section with the definition of finite determinacy.

Definition 3.1. Given $f \in \mathscr{E}(n, p)$ and $k \in \mathbb{N}$, we say that f is *k-determined* if for any $g \in \mathscr{E}(n, p)$ such that $j^k f(0) = j^k g(0)$, then f, g are $\mathscr{A}$-equivalent. We say that f is *finitely determined* (FD) if it is k-determined for some $k \in \mathbb{N}$.

From the definition we deduce that if f is k-determined then f is $\mathscr{A}$-equivalent to $j^k f(0)$. Thus, when studying FD germs, we can assume without loss of generality that f is the germ of a polynomial mapping. Another consequence of the definition and of Proposition 2.13 is that if $f \in \mathscr{E}(n, p)$ is stable, then it is $(p + 1)$-determined.

Finite determinacy is a very desirable property, but usually it is difficult to check it directly from the defintion. By this reason, the criteria of finite determinacy are very important. The following criterion is known as the infinitesimal criterion of finite determinacy. It is due to J. Mather and a proof can be found in [42, 1.2].

Theorem 3.2. *A germ $f \in \mathscr{E}(n, p)$ is FD if and only if its $\mathscr{A}_e$-codimension is finite.*

Next property is analogous to the Proposition 2.23 and it relates the finite determinacy of the three classes of map germs: C^∞, real analytic and complex analytic. The proof is based again on the fact that the $\mathscr{A}_e$-codimension coincides in the three classes [42, 1.7].

Proposition 3.3. *Let $f \in \mathscr{E}(n, p)$ a real analytic germ, then the following statements are equivalent:*

(1) *f is FD as a C^∞ germ.*
(2) *f is FD as a real analytic germ.*
(3) *The complexification $\hat{f}$ is FD as a complex analytic germ.*

We give now the geometric criterion of finite determinacy of Mather-Gaffney which works for complex analytic germs. Roughly speaking, it says that a germ is FD if and only if it has isolated instability at the origin. The proof can be found in [42, Theorem 2.1].

Theorem 3.4. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ a complex analytic germ. Then f is FD if and only if there exists a representative $f : U \to V$ where U, V are open neighbourhoods of the origin in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively, such that $f^{-1}(0) \cap \Sigma(f) = \{0\}$ and the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is a locally stable mapping.*

If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is FD and is defined by polynomials, then we can complexify $\hat{f} : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and apply the geometric criterion to $\hat{f}$. We deduce that there exists a representative $f : U \to V$ where U, V open neighbourhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that $f^{-1}(0) \cap \Sigma(f) = \{0\}$ and the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is a locally stable mapping.

The converse is not true in general in the real case. For instance, consider the function $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ given by $f(x, y) = (x^2 + y^2)^2$. We have $f^{-1}(0) = \Sigma(f) = \{0\}$ and the restriction $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is regular and hence, locally stable. However, $\hat{f}^{-1}(0) = \Sigma(\hat{f}) = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$, so f is not FD by 3.3 and 3.4.

Since $f^{-1}(0) \cap \Sigma(f) = \{0\}$, after shrinking the neighbourhoods U, V if necessary, we can assume that the restriction $f : \Sigma(f) \to V$ is finite. Moreover, if $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is a locally stable mapping, then the 0-stable types are isolated points in $U \setminus \{0\}$. But since these sets are semialgebraic, then by the Curve Selection Lemma [28], we have that the 0-stable types are also isolated points in U. Thus, we can shrink the neighbourhoods U, V in such a way that f has no 0-stable singularities in $U \setminus \{0\}$.

This fact motivates the following definition.

Definition 3.5. We say that a germ $f \in \mathscr{E}(n, p)$ has *isolated instability* (II) if there exists a representative $f : U \to V$ where U, V are open neighbourhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that

(1) $f^{-1}(0) \cap \Sigma(f) = \{0\}$,
(2) the restriction $f : \Sigma(f) \to V$ is finite,
(3) the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is a locally stable mapping with no 0-stable singularities.

In such case we also say that $f : U \to V$ is a *good representative* of f. In the case that f is a polynomial mapping, we also add the condition that the open sets U, V are semialgebraic.

It follows from the above remarks that any FD germ has II, but the converse is not true in general.

Other important definitions related to the finite determinacy are the concepts of finite type singularity and of finite germ. These two concepts correspond to the finite determinacy when we consider the groups $\mathscr{K}$ and $\mathscr{C}$ respectively instead of the group $\mathscr{A}$ (see [42, Theorem 1.2]).

Definition 3.6. Given $f \in \mathscr{E}(n,p)$, we say that f has *finite singularity type* if

$$\dim_{\mathbb{R}} \frac{\theta(f)}{T\mathscr{K}_e f} < +\infty,$$

where $T\mathscr{K}_e f$ was defined in 2.8. We say that f is *finite* if

$$\dim_{\mathbb{R}} Q(f) < +\infty.$$

Some properties can be deduced immediately from the definitions:

(1) f is FD $\Longrightarrow f$ has finite singularity type.
(2) f is finite $\Longrightarrow f$ has finite singularity type.
(3) f is finite $\Longrightarrow n \leq p$.
(4) If $n \leq p$, f has finite singularity type $\Longrightarrow f$ is finite.

Properties (1) and (2) are consequence of the fact that the $\mathscr{C}$ and the $\mathscr{A}$-equivalence imply the $\mathscr{K}$-equivalence. Property (3) follows from the fact that if $n > p$, then $f^*\mathfrak{m}_p$ is generated by less that n elements and hence, it cannot have finite codimension. Finally, property (4) is proved in [42, 2.4.(ii)] (note that the case $n < p$ is trivial).

Given a complex analytic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, we have that f has finite singularity type if and only if $f^{-1}(0) \cap \Sigma(f) = \{0\}$ and f is finite if and only if $f^{-1}(0) = \{0\}$. Both properties are consequence of the Hilbert Nullstellensatz (in the complex analytic version [15, Theorem 3.4.4]).

In the real case we have only one of the implications: if $f \in \mathscr{E}(n,p)$ has finite singularity type then $f^{-1}(0) \cap \Sigma(f) = \{0\}$ and if f is finite then $f^{-1}(0) = \{0\}$. Another two very important properties are stated in the next theorem, the proof can be found in [11, 2.8, 3.1].

Theorem 3.7. *Let $f \in \mathscr{E}(n,p)$.*

(1) *f has finite singularity type if and only if there exists a stable unfolding F of f.*
(2) *If F, G are r-parameter stable unfoldings of f, then F, G are $\mathscr{A}$-equivalent.*

Let $f \in \mathscr{E}(n,p)$ be of finite singularity type and let F a stable unfolding of f. Given a stable type represented by the $\mathscr{A}$-class of a stable multi-germ $g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, we say that F presents the stable type if for any representative $F : U \to V$ there exists $(u; y') \in V$ such that the multi-germ $f_u : (\mathbb{R}^n, S') \to (\mathbb{R}^p, y')$, with $S' = f_u^{-1}(y') \cap \Sigma(f_u)$, is $\mathscr{A}$-equivalent to g.

Definition 3.8. We say that $f \in \mathscr{E}(n,p)$ has *discrete stable type* (DST) if there exists a stable unfolding F of f which only presents a finite number of stable types.

Some cases in which $f \in \mathscr{E}(n,p)$ has DST are:

(1) when (n,p) are nice dimensions or are in the boundary of the nice dimensions in Mather's sense [21];
(2) when f has corank 1.

Definition 3.9. Let $f : U \to V$ be a good representative of a germ $f \in \mathscr{E}(n,p)$ with II and DST. We construct a stratification $(\mathcal{A},\mathcal{B})$ of f defined as follows:

- The strata B of $\mathcal{B}$ are either $B = \{0\}$, $B = V \setminus \Delta(f)$ or B is the analytic stratum in the target of $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ for some $y \in \Delta(f)$ and $S = f^{-1}(y) \cap \Sigma(f)$.
- The strata A of $\mathcal{A}$ are either strata of the form $A = f^{-1}(B) \cap \Sigma(f)$ or strata of the form $A = f^{-1}(B) \setminus \Sigma(f)$, for some $B \in \mathcal{B}$. In particular, we always have the strata $A = \{0\}$ and $A = f^{-1}(0) \setminus \{0\}$ (if $f^{-1}(0) \neq \{0\}$).

We call $(\mathcal{A},\mathcal{B})$ the *stratification by stable types.* The fact that f has DST guarantees that the stratification is finite. If in addition f is polynomial, then all the strata are semialgebraic sets.

4. The cone structure theorem for map germs with isolated zeros

In this section, we show the cone structure theorem for FD germs $f \in \mathscr{E}(n,p)$, with $f^{-1}(0) = \{0\}$ and DST, following the arguments of Fukuda in [8]. We fix some notation:

$$D^p_\epsilon = \{y \in \mathbb{R}^p : \|y\|^2 \leq \epsilon\}, \quad S^{p-1}_\epsilon = \{y \in \mathbb{R}^p : \|y\|^2 = \epsilon\}.$$

Given a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ we take a representative $f : U \to V$ and put:

$$\tilde{D}^n_\epsilon = f^{-1}(D^p_\epsilon), \quad \tilde{S}^{n-1}_\epsilon = f^{-1}(S^{p-1}_\epsilon).$$

We recall that if $f \in \mathscr{E}(n,p)$ is FD, then after coordinate changes we can assume that it is polynomial and has II.

Theorem 4.1 ([8]). *Let $f : U \to V$ a good representative of a polynomial map germ $f \in \mathscr{E}(n,p)$ with II, DST and such that $f^{-1}(0) = \{0\}$. Then, there exists $\epsilon_0 > 0$ such that for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\tilde{S}^{n-1}_\epsilon$ is a smooth submanifold diffeomorphic to S^{n-1},*
2. *$f|_{\tilde{S}^{n-1}_\epsilon} : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is a stable mapping, whose $\mathscr{A}$-class is independent of ϵ,*
3. *$f|_{\tilde{D}^n_\epsilon \setminus \{0\}} : \tilde{D}^n_\epsilon \setminus \{0\} \to D^p_\epsilon \setminus \{0\}$ is $\mathscr{A}$-equivalent to the product map $\mathrm{id} \times f|_{\tilde{S}^{n-1}_\epsilon} : (0,\epsilon] \times \tilde{S}^{n-1}_\epsilon \to (0,\epsilon] \times S^{p-1}_\epsilon$,*
4. *By adding the origin, $f|_{\tilde{D}^n_\epsilon} : \tilde{D}^n_\epsilon \to D^p_\epsilon$ is C^0-$\mathscr{A}$-equivalent to the cone of $f|_{\tilde{S}^{n-1}_\epsilon}$.*

Proof. Let $(\mathcal{A},\mathcal{B})$ be the stratification by stable types of $f : U \to V$, which has a finite number of semialgebraic strata. We consider the polynomial function $g : U \to \mathbb{R}$ given by $g = \|f\|^2$ and its restriction $g|_{A_i} : A_i \to \mathbb{R}$ to each stratum A_i of $\mathcal{A}$. By the Curve Selection Lemma [28], each $g|_{A_i}$ has a finite number of critical values. Thus, there exists $\epsilon_0 > 0$ such that for any ϵ with $0 < \epsilon \leq \epsilon_0$, ϵ is a regular value of g and $g|_{A_i}$ for all $A_i \in \mathcal{A}$.

Since ϵ is a regular value of g, $\tilde{S}^{n-1}_\epsilon = g^{-1}(\epsilon)$ is a hypersurface in U. Moreover, the condition that ϵ is a regular value of $g|_{A_i}$, for all $A_i \in \mathcal{A}$, is equivalent to that S^{p-1}_ϵ is transverse to all the strata B_i of $\mathcal{B}$. By Proposition

2.22, the restriction $f|_{\tilde{S}^{n-1}_\epsilon}:\tilde{S}^{n-1}_\epsilon\to S^{p-1}_\epsilon$ is stable. Thus, we have showed the first part of (2).

To see (1), we use Reeb's theorem [26, pag. 25]. Since $f^{-1}(0)=\{0\}$, 0 is an isolated minimum of g. Then, $\tilde{D}^n_\epsilon=g^{-1}([0,\epsilon])$, is homeomorphic to the closed disk D^n. Thus, $\tilde{S}^{n-1}_\epsilon=\partial\tilde{D}^n_\epsilon$ is homeomorphic (and hence diffeomorphic) to S^{n-1}.

It only remains to show the second part of (2) and (3), since (4) is an immediate consequence of (3). We set $I=(0,\epsilon]$ and consider the following diffeomorphisms:

$$\begin{aligned}\Phi:\tilde{D}^n_\epsilon\setminus\{0\}&\longrightarrow I\times\tilde{S}^{n-1}_\epsilon, & \Psi: D^p_\epsilon\setminus\{0\}&\longrightarrow I\times S^{p-1}_\epsilon,\\ x&\longmapsto(g(x),\phi(x)), & y&\longmapsto(\|y\|^2,\sqrt{\epsilon}\frac{y}{\|y\|}),\end{aligned}$$

where $\phi(x)$ is the point of $\tilde{S}^{n-1}_\epsilon$ where the integral curve of the gradient of g passing through x meets $\tilde{S}^{n-1}_\epsilon$. We define $F:I\times\tilde{S}^{n-1}_\epsilon\to I\times S^{p-1}_\epsilon$ as $F=\Psi\circ f\circ\Phi^{-1}$. By construction, we have that $F(\{t\}\times\tilde{S}^{n-1}_\epsilon)\subset\{t\}\times S^{p-1}_\epsilon$, for any $t\in I$. This implies that F can be written in the form $F(t;x)=(t;f_t(x))$, with $f_t:\tilde{S}^{n-1}_\epsilon\to S^{p-1}_\epsilon$ and $t\in I$.

It is obvious that f_t is $\mathscr{A}$-equivalent to $f|_{\tilde{S}^{n-1}_t}$ and thus, f_t is stable. In particular, the unfolding F must be trivial, that is, there exist diffeomorphisms $H:I\times\tilde{S}^{n-1}_\epsilon\to\tilde{I}\times S^{n-1}_\epsilon$ and $K:I\times S^{p-1}_\epsilon\to I\times S^{p-1}_\epsilon$ of the form $H(t;x)=(t;h_t(x))$ and $K(t;y)=(t;k_t(y))$ and such that $K\circ F\circ H^{-1}=\mathrm{id}\times f_\epsilon$. Hence, we have (3). The second part of (2) follows form the fact that $k_t\circ f_t\circ h_t^{-1}=f_\epsilon$. □

Definition 4.2. Let $f:U\to V$ a good representative of a polynomial map germ $f\in\mathscr{E}(n,p)$ with II, DST and such that $f^{-1}(0)=\{0\}$. We say that $\epsilon_0>0$ is a *Milnor-Fukuda radius* for f if for any ϵ with $0<\epsilon\le\epsilon_0$, S^{p-1}_ϵ is transverse to the stratification by stable types of f.

We also say that the mapping $f|_{\tilde{S}^{n-1}_\epsilon}:\tilde{S}^{n-1}_\epsilon\to S^{p-1}_\epsilon$ is the *link* of f and denote it by $L(f)$. It follows from Theorem 4.1 that:

(1) the link is a stable mapping between spheres,
(2) the link is well defined up to $\mathscr{A}$-equivalence,
(3) the germ f is C^0-$\mathscr{A}$-equivalent to the cone of its link.

We include now a couple of important remarks with respect to Theorem 4.1 and the definition of the link.

Remark 4.3. The condition $f^{-1}(0)=\{0\}$ is always satisfied if $f\in\mathscr{E}(n,p)$ is FD and $n\le p$. In the case $n>p$, we may have the two possibilities, either $f^{-1}(0)=\{0\}$ or $f^{-1}(0)\ne\{0\}$. We will give another version of the cone structure theorem for the case $f^{-1}(0)\ne\{0\}$ in the last section.

Remark 4.4. If f is real analytic instead of polynomial, then the theorem is still valid, it is enough to use the semianalytic version of the Curve Selection Lemma. When f is only of class C^∞, if f is FD, f is $\mathscr{A}$-equivalent to a polynomial map and hence, the theorem is valid for a representative of a germ which is $\mathscr{A}$-equivalent to f. If we want to apply the theorem directly to a representative $f:U\to V$ of f, then we have to change the spheres

S^{p-1}_ϵ by hypersurfaces $P_\epsilon \subset V$ diffeomorphic to the sphere S^{p-1} (since the diffeomorphisms do not preserve spheres in general).

More exactly, there exists a function called control function $\rho : V \to \mathbb{R}$ with a unique critical point of Morse type in the origin, which plays the role of the function $\|y\|^2$ in the analytic case. We consider $g = \rho \circ f : U \to \mathbb{R}$ and choose $\epsilon_0 > 0$ in such a way that for all ϵ with $0 < \epsilon \leq \epsilon_0$, ϵ is a regular value of $g|_{A_i}$ for all $A_i \in \mathcal{A}$. The hypersurfaces P_ϵ are defined as $P_\epsilon = \rho^{-1}(\epsilon)$ and are diffeomorphic to S^{p-1}. Then, the inverse image $N_\epsilon = f^{-1}(P_\epsilon)$ is diffeomorphic to S^{n-1}, the restriction $f|_{N_\epsilon} : N_\epsilon \to P_\epsilon$ is stable and f is topologically equivalent to the cone of $f|_{N_\epsilon}$.

Remark 4.5. If f has no DST, then the theorem is still valid with the only difference that the link $f|_{\tilde{S}^{n-1}_\epsilon} : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is C^0-stable instead of stable. The proof in this case can be adapted by using the Mather canonical stratification (see [11]) instead of the stratification by stable types. We leave the details of this construction for the reader.

Next corollary is an immediate consequence of Theorem 4.1.

Corollary 4.6. *Let $f, g \in \mathscr{E}(n,p)$ be two FD germs with $f^{-1}(0) = g^{-1}(0) = \{0\}$. If $L(f), L(g)$ are C^0-$\mathscr{A}$-equivalent, then f, g are C^0-$\mathscr{A}$-equivalent.*

Exemple 4.7. Let $f \in \mathscr{E}(1,1)$ be a FD germ, then the link is a mapping $\gamma : S^0 \to S^0$. Since that $S^0 = \{-1, 1\}$, we have only two non equivalent possibilities, namely, either $\gamma = \text{id}$ or $\gamma = \text{constant}$. In fact, if f is FD, then the infinite jet $j^\infty f(0) \neq 0$. Thus, we have that $j^\infty f(0) = a_k x^k + \dots$, with $a_k \neq 0$, and f is $\mathscr{A}$-equivalent to x^k. We have

$$\gamma = \begin{cases} \text{id}, & \text{if } k \text{ is odd}, \\ \text{constant}, & \text{if } k \text{ is even}. \end{cases}$$

Basically, this is the well known criterion by the Calculus students for the existence of local maxima, minima or inflections in one variable functions (fig. 4).

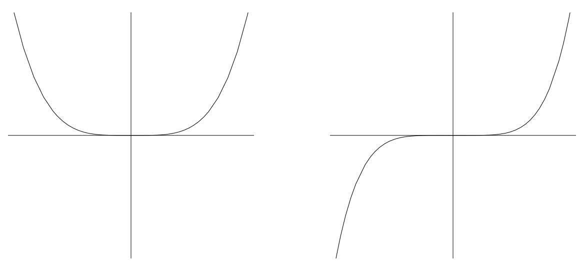


FIGURE 4. Graph of $f \in \mathscr{E}(1,1)$ when k even (left) and k odd (right)

Exemple 4.8. Given a FD germ $f \in \mathscr{E}(1,2)$, its link is a non constant mapping $\gamma : S^0 \to S^1$. In this case, two non constant mappings $\gamma_1, \gamma_2 : S^0 \to S^1$ are always C^0-$\mathscr{A}$-equivalent, it is enough to take any homeomorphism from S^1 to S^1 which takes two points in other two points. As a consequence, there exists a unique topological class of FD germs $f \in \mathscr{E}(1,2)$ (fig. 5).

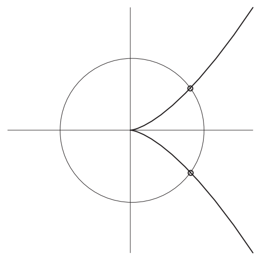


FIGURE 5. The link of a FD germ $f \in \mathscr{E}(1,2)$

Exemple 4.9. Given a FD germ $f \in \mathscr{E}(2,1)$ such that $f^{-1}(0) = \{0\}$, the link is a constant mapping $\gamma : S^1 \to S^0$ and again we have only one topological class (fig. 6).

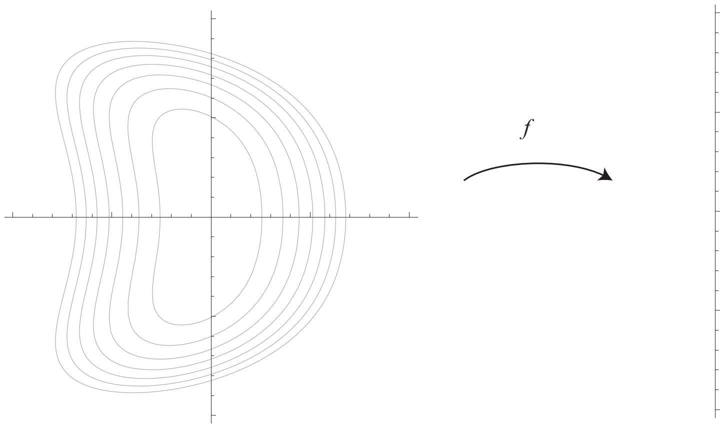


FIGURE 6. A FD germ $f \in \mathscr{E}(2,1)$ with $f^{-1}(0) = \{0\}$

We conclude this section with the main open questions related to the topological classification of FD germs $f \in \mathscr{E}(n,p)$ with isolated zeros:

(1) Find a good combinatorial model which codifies all the topological information of a stable mapping $\gamma : S^{n-1} \to S^{p-1}$ (and hence, of the germ f).

(2) Determine the stable mappings $\gamma : S^{n-1} \to S^{p-1}$ which can be realized as the link of a FD germ f, with $f^{-1}(0) = \{0\}$.

(3) Determine if the converse of Corollary 4.6 is true or not, that is, if f, g are C^0-$\mathscr{A}$-equivalent, then does this imply that $L(f), L(g)$ are C^0-$\mathscr{A}$-equivalent?

(4) Find relations between analytic invariants of f (corank, 0-stable invariants, $\mathscr{A}_e$-codimension, etc.) and the topological invariants of the link (number of 0-stable singularities, Vassiliev invariants, etc.).

(5) Study the topological transitions in 1-parameter families of FD germs, in particular, study the topological triviality of the family.

5. Gauss words

Let $f \in \mathscr{E}(2,3)$ be FD germ. Then the link is a stable map $\gamma : S^1 \to S^2$, that is, γ defines a closed regular curve in S^2 with only transverse double points or crossings. We call such type of curves *doodles*. The topological classification of doodles in the sphere S^2 (or in the plane $\mathbb{R}^2$) is well known since Gauss time [10]. The combinatorial model is given by the so-called "Gauss words". Most the results of this section appear in the paper [22].

Definition 5.1. Let $\gamma : S^1 \to S^2$ be a doodle with r crossings. We choose r letters $a_1, \dots, a_r$ to label the crossings, orientations in S^1 and S^2, and a base point $z_0 \in S^1$. We define the *Gauss word* as the sequence of crossings starting from the base point and following the orientation of the curve. Each letter a_i appears twice, one with exponent $+1$ and another one with exponent -1, according to the orientation of the two branches near the crossing in the sphere S^2 (see fig. 7).

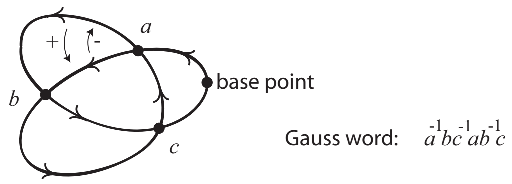


Figure 7. Gauss word of the trefoil

It is obvious that the Gauss word is not uniquely defined since it depends on the choice of the labels of the crossings, the base point and the orientations in S^1, S^2. Different choices will produce the following changes in the Gauss word:

(1) permuting the alphabet set $a_1, \dots, a_r$;
(2) cyclically permuting the sequence;
(3) reversing the sequence;
(4) changing all the exponents from $+1$ to -1 and vice versa.

We say that two Gauss words are *equivalent* if they related by means of these four operations. Up to this equivalence, the Gauss word is now well defined. Moreover, the following theorem shows that the Gauss words provide a complete invariant in the topological classification of doodles in the sphere.

Theorem 5.2 (Gauss Theorem)**.** *Two doodles on the sphere are C^0-$\mathscr{A}$-equivalent if and only if their Gauss words are equivalent.*

Proof. Let $\gamma, \delta : S^1 \to S^2$ be two doodles which are C^0-$\mathscr{A}$-equivalent. There exist homeomorphisms $\alpha : S^1 \to S^1$ and $\beta : S^2 \to S^2$ such that $\delta = \beta \circ \gamma \circ \alpha^{-1}$. We start with γ and we choose letters $a_1, \dots, a_r$ to label the crossings, a base point $z_0 \in S^1$ and orientations in S^1, S^2, so that we have the Gauss word of γ. Since β takes crossings of γ into crossings of δ, we can choose for each crossing of δ the same letter of the corresponding crossing in γ through β. We also take $\alpha(z_0) \in S^1$ as the base point of δ. Finally, we choose in S^1, S^2 the orientations induced by α, β respectively. With these choices, we have that the Gauss word of δ is equal to the Gauss word of γ.

To see the converse, we first observe that each doodle $\gamma : S^1 \to S^2$ has a natural CW-structure: in S^1 the 0-cells are the inverse images of the crossings and the 1-cells are the connected components of the complement. In S^2, the 0-cells are the crossings, the 1-cells are the edges of the curve joining the crossings and the 2-cells are the connected components of the complement of the curve (this is possible because the curve is a connected graph).

It follows that the CW-structure of S^2 can be read from the Gauss word. In fact, the 0-cells are given by the letters $a_1, \dots, a_r$, each 1-cell is an oriented edge defined by two consecutive letters $a_i^\epsilon a_j^\eta$ in the Gauss word (including the oriented edge joining the last with the first letter) and each 2-cell is a face which is determined by a closed sequence of oriented edges or their inverses.

Assume now that $\gamma, \delta : S^1 \to S^2$ have the same Gauss word. Then the two S^2 are isomorphic as CW-complexes with the CW-structure induced by γ, δ. We choose any cellular homeomorphism $\beta : S^2 \to S^2$. Then we construct another cellular homeomorphism $\alpha : S^1 \to S^1$ such that $\delta = \beta \circ \gamma \circ \alpha^{-1}$. In fact, on each 1-cell E, α is univocally defined as $\alpha|_E = (\delta^{-1} \circ \beta \circ \gamma)|_E$ and then α is extended by continuity to the 0-cells.

If $\gamma, \delta : S^1 \to S^2$ have equivalent Gauss words, then we can take homeomorphisms $\alpha : S^1 \to S^1$ and $\beta : S^2 \to S^2$ such that $\beta \circ \gamma \circ \alpha^{-1}$ and δ have the same Gauss word. Then, we apply the above argument to these two doodles. $\square$

Exemple 5.3. In the trefoil (see fig.7), the CW-structure on the sphere is constructed from the Gauss word $a^{-1}bc^{-1}ab^{-1}c$ as follows:

(1) we have three 0-cells given by a, b and c;
(2) we have six 1-cells given by $a^{-1}b$, bc^{-1}, $c^{-1}a$, ab^{-1}, $b^{-1}c$ and ca^{-1};
(3) there are five 2-cells given by three 2-gons $\{ab^{-1}, ba^{-1}\}$, $\{bc^{-1}, cb^{-1}\}$, $\{ca^{-1}, ac^{-1}\}$ and two triangles $\{a^{-1}b, b^{-1}c, c^{-1}a\}$, $\{a^{-1}c, c^{-1}b, b^{-1}a\}$.

The theorem is not true for doodles in the plane $\mathbb{R}^2$. For instance, the two doodles in fig. 8 are topologically equivalent on the sphere and have the same Gauss word aa^{-1}, but they are not topologically equivalent on the plane (in fact, they have different Whitney index).

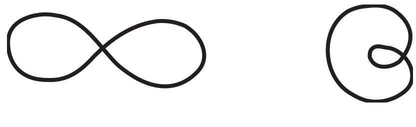


FIGURE 8. Two non equivalent doodles in the plane with the same Gauss word aa^{-1}

We show in fig. 9 the classification of doodles in the sphere with up to three crossings. There are 10 non equivalent doodles and their corresponding Gauss words are the following;

(a) $\emptyset$

(b) aa^{-1}
(c) $ab^{-1}ba^{-1}$
(d) $abb^{-1}a^{-1}$
(e) $ab^{-1}cc^{-1}ba^{-1}$
(f) $ab^{-1}c^{-1}cba^{-1}$
(g) $abcc^{-1}b^{-1}a^{-1}$
(h) $ab^{-1}ca^{-1}bc^{-1}$
(i) $aa^{-1}bb^{-1}cc^{-1}$
(j) $aa^{-1}b^{-1}bcc^{-1}$

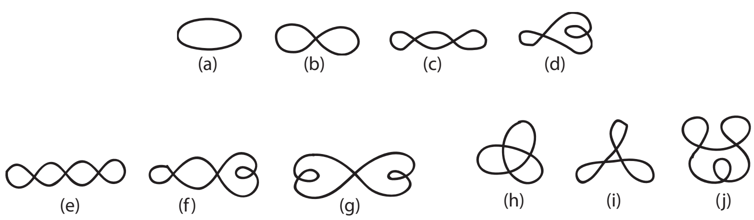


Figure 9. Doodles with up to three crossings

Gauss was interested in the problem of "planarity" of Gauss words: determine the words which can be realized as the word of a doodle in the sphere (or in the plane). It is well known that any Gauss word can be realized as the word of a doodle in some orientable compact surface of genus g. For instance, the word $aba^{-1}b^{-1}$ cannot be realized in the sphere (or the plane), but it can be realized in the torus (see fig. 10).

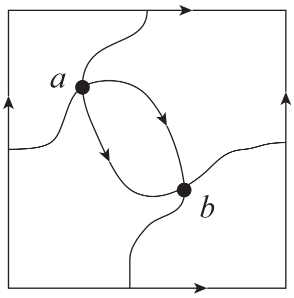


Figure 10. A doodle in the torus with Gauss word $aba^{-1}b^{-1}$

Gauss could not solve the planarity problem, but he only was able to find a necessary condition. The planarity problem was completely solved by M. Dehn in 1936 [6]. The planarity problem of Gauss words is of the same nature as the planarity problem of graphs (Kuratowski Theorem). Nowadays, the Gauss words constitute a very active field of research in Computational Geometry.

Definition 5.4. Given a FD germ $f \in \mathscr{E}(2,3)$, we define the *Gauss word* of f as the Gauss word of the doodle of f.

It follows from Gauss Theorem that if two map germs have equivalent Gauss words, then they are C^0-$\mathscr{A}$-equivalent. We will see that the converse is also true. But to to this we need to analyze the structure of a FD germ.

We begin with the characterization of stable singularities. We see that a C^∞ mapping $f : N^2 \to P^3$ is stable if and only if it is *semiregular* in the sense of Whitney [45]: f is an immersion with normal crossings, except at isolated points, where f presents singularities of type *cross-cap* or *Whitney umbrella.* At each of this points, the germ of f is $\mathscr{A}$-equivalent to the germ in $\mathscr{E}(2,3)$ given by $(x,y) \mapsto (x,y^2,xy)$ (see fig. 11).

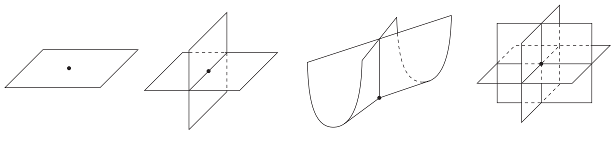


FIGURE 11. Stable singularities of surfaces in $\mathbb{R}^3$

Theorem 5.5. *The only stable multi-germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ are: regular simple point, transverse double point, transverse triple point and cross-cap.*

Proof. We first show that a singular germ $f \in \mathscr{E}(2,3)$ is stable if and only if it has cross-cap type. After coordinate changes in the source and the target, we can assume that f is given by the standard parametrization $f(x,y) = (x,y^2,xy)$, then:

$$\begin{aligned} T\mathscr{K}_e f &= \mathscr{E}_2 \left\{ \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 2y \\ x \end{pmatrix} \right\} + \langle x, y^2 \rangle \mathscr{E}_2^2 \\ &= \mathscr{E}_2 \left\{ \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \right\}. \end{aligned}$$

We have that $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the classes of the canonical basis $\{e_1, e_2, e_3\}$, hence $\overline{\omega} f$ is surjective and f is stable by Lemma 2.9.

To see the converse, suppose first that f is stable and has rank 0. Then $T\mathscr{K}_e f \subset \mathfrak{m}_2\theta(f)$. Since $\theta(f)/\mathfrak{m}_2\theta(f)$ has dimension 3, we must have necessarily that $T\mathscr{K}_e f = \mathfrak{m}_2\theta(f)$. Moreover, $(f^*\mathfrak{m}_3) \subset \mathfrak{m}_2^2\theta(f)$, hence the classes of $\partial f/\partial x$ and $\partial f/\partial y$ should generate $\mathfrak{m}_2\theta(f)/\mathfrak{m}_2^2\theta(f)$ over $\mathbb{R}$. But this is not possible, since this space has dimension 6.

Thus, if f is stable, it must have rank 1 and after a coordinate change in the source, we can assume that $f(x,y) = (x, g(x,y))$, for some germ $g \in \mathscr{E}(2,2)$. In other words, we see f as an unfolding of $g_0(y) = g(0,y)$. In particular, we have:

$$\frac{\theta(f)}{T\mathscr{K}_e(f)} \cong \frac{\theta(g_0)}{T\mathscr{K}_e(g_0)} \cong \frac{\mathscr{E}_1^2}{\langle g_0' \rangle}.$$

If $g_0 \in \mathfrak{m}_1^3\mathscr{E}_1^2$, then $g_0' \in \mathfrak{m}_1^2\mathscr{E}_1^2$ and thus $\dim_{\mathbb{R}}(\mathscr{E}_1^2/\langle g_0' \rangle) \geq 4$, which is not possible by the surjectivity of $\overline{\omega} f$. Hence, g_0 must have order 2. But this implies that $Q(f) \cong \mathscr{E}_1/\langle y^2 \rangle$, hence f is $\mathscr{A}$-equivalent to the cross-cap by Theorem 2.11.

We consider now multi-germs $f : (\mathbb{R}^2, S) \to (\mathbb{R}^3, y)$, with $S \subset \mathbb{R}^2$ a finite set. If one of the points $x_i \in S$ is singular, then f has cross-cap type at x_i and the analytic stratum is only the point $\{x_i\}$. Thus, the regular intersection condition of Theorem 2.15 implies that $S = \{x_i\}$. Otherwise, if all the points of S are regular, then f is an immersion with normal crossings and we find the remaining types: regular simple point, transverse double point and transverse triple point. □

Assume now that $f \in \mathscr{E}(2, 3)$ is FD. The 0-stable types are the cross-caps and the triple points. Thus, a good representative of f is a mapping $f : U \to V$ where $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$ are open neighbourhoods of the origin such that:

(1) $f^{-1}(0) = \{0\}$,
(2) $f : U \to V$ is proper,
(3) $f : U \setminus \{0\} \to V \setminus \{0\}$ is an immersion with only transverse double points.

An important set associated with f is the *double point curve*, which is defined as

$$D(f) = \{z \in U : f^{-1}(f(z)) \neq \{z\}\} \cup S(f),$$

where $S(f)$ is the singular set. Then $D(f)$ is a closed subset of U. Since f is a good representative, it follows that $S(f) = \{0\}$ and that $D(f) \setminus \{0\}$ is a 1-dimensional submanifold of U. By shrinking the neighbourhoods if necessary, we can assume that all the connected components of $D(f) \setminus \{0\}$ are arcs going from the origin to the boundary of U.

Moreover, f restricted to each connected component is a diffeomorphism, so that the image $f(D(f)) \setminus \{0\}$ is also a 1-dimensional submanifold of V, whose connected components are arcs going from the origin to the boundary of V (since f is proper). Moreover, the restriction $f : D(f)\setminus\{0\} \to f(D(f))\setminus \{0\}$ is a 2-fold covering. The connected components of $D(f)$ (resp. $f(D(f))$) are called *half-branches* of $D(f)$ (resp. $f(D(f))$).

We claim that we can recover the Gauss word of f just by looking at the relative position of the half-branches of $D(f)$ and $f(D(f))$ and the orientation of the leaves of $f(U)$ at each half-branch. In fact, each half-branch of $f(D(f))$ corresponds to a crossing in the doodle of f. So, we can choose letters $a_1, \dots, a_r$ to label the half-branches. We also choose orientations in U, V and a base point in U. Then, we construct the Gauss word as the sequence of letters according to the relative position of the half-branches of $D(f)$ in U, starting from the base point and following the orientation in U. Moreover, we put the exponent $+1$ if the two leaves of $f(U)$ intersect positively along the half-branch or -1 otherwise. It is obvious that the word obtained with this method is exactly the Gauss word of f (see fig. 12).

Assume now that we have two FD $f, g \in \mathscr{E}(2, 3)$ which are C^0-$\mathscr{A}$-equivalent. Then, the homeomorphisms must preserve the double point sets $D(f)$ and $f(D(f))$. An argument analogous to that of the proof of Theorem 5.2 gives that f, g have the same Gauss word (up to equivalence). Thus, we have proved the following theorem (see [22, Corollaries 3.4 and 3.8]).

Theorem 5.6. *Let $f, g \in \mathscr{E}(2, 3)$ be two FD germs. The following statements are equivalent:*

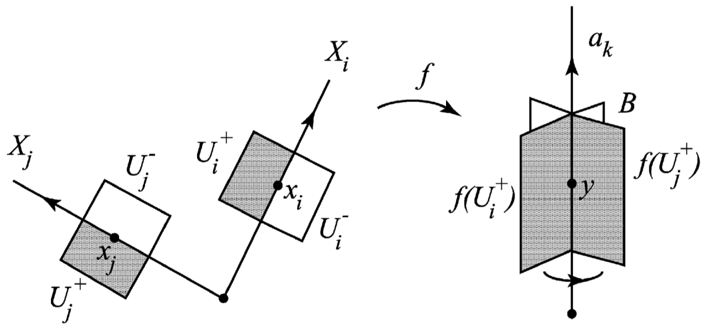


FIGURE 12. Orientation of the branches

(1) *f, g are C^0-$\mathscr{A}$-equivalent,*
(2) *the doodles of f, g are C^0-$\mathscr{A}$-equivalent,*
(3) *f, g have equivalent Gauss words.*

Exemple 5.7. All the doodles with up to three crossings (see fig. 9) are realizable as the link of a FD map germ $f \in \mathscr{E}(2,3)$:

(a) $(x, y, 0)$,
(b) (x, y^2, xy),
(c) $(x, y^2, y(x^2 - y^2))$,
(d) $(x, xy + y^3, xy^3 + \frac{3}{2}y^5)$,
(e) $(x, y^2, xy(x^2 - y^2))$,
(f) $(x, x^4 - 6x^2y^2 + y^4, x^3y - xy^3)$,
(g) $(x, x^4 - 6x^2y^2 + y^4, x^3y - xy^3)$,
(h) $(x, xy + y^3, xy^2 + \frac{3}{4}y^4)$,
(i) $(x, xy + y^3, xy^2 + \frac{5}{4}y^4)$,
(j) $(x^2, xy + y^3, \frac{1}{2}x^3 + \frac{1}{4}x^2y + 3xy^3 + 3y^5)$.

To check this, we use a tailor-made computer program `SphereXSurface` by A. Montesinos-Amilibia [30], which pictures the doodle of any map. We remark that all of them except "Mickey" (j) admit a corank 1 realization. We do not know, up to now, if it is also possible to find a corank 1 realization for this doodle.

To finish this section, we see the topological classification of all FD germs $f \in \mathscr{E}(2,3)$ with Boardman type $\Sigma^{1,0}$. We recall the definition of Boardman symbol of order 2.

Definition 5.8. Given $f \in \mathscr{E}(n,p)$, let $M_1, \dots, M_r$ be the minors of order $n - i + 1$ of the Jacobian matrix of f and set $\tilde{f} = (f_1, \dots, f_p, M_1, \dots, M_r)$. We say that it has *Boardman type* $\Sigma^{i,j}$ if

$$\dim_{\mathbb{R}} \ker df(0) = i, \quad \dim_{\mathbb{R}} \ker d\tilde{f}(0) = j.$$

The following lemma is due to Mond [29] and gives a prenormal form for all germs with Boardman type $\Sigma^{1,0}$.

Lemma 5.9. *Let $f \in \mathscr{E}(2,3)$ be a germ with Boardman type $\Sigma^{1,0}$. Then f is $\mathscr{A}$-equivalent to a map germ of the form*

$$\tilde{f}(x, y) = (x, y^2, yp(x, y^2)),$$

for some $p \in \mathscr{E}_2$.

Proof. The condition that $\dim_{\mathbb{R}} \ker df(0) = 1$ implies that f has corank 1, then after $\mathscr{A}$-equivalence, f can be written in the form

$$f(x, y) = (x, g(x, y), h(x, y)),$$

for some $g, h \in \mathfrak{m}_2^2$. Then the 2-minors of the Jacobian matrix are g_y, h_y, $g_x h_y - g_y h_x$, where the subscripts mean the partial derivatives. Then, an easy computation shows that f has Boardman type $\Sigma^{1,0}$ if and only if either $g_{yy}(0) \neq 0$ or $h_{yy}(0) \neq 0$.

Assume, for instance, that $g_{yy}(0) \neq 0$. Then, we can write

$$f(x, y) = (x, ax^2 + 2bxy + cy^2 + \tilde{g}(x, y), h(x, y)),$$

where $\tilde{g} \in \mathfrak{m}_2^3$ and $c \neq 0$. If $c > 0$, we put

$$ax^2 + 2bxy + cy^2 = (\frac{b}{\sqrt{c}}x + \sqrt{c}y)^2 + (a - \frac{b^2}{c})x^2,$$

then the coordinate change in the source given by $\bar{y} = (b/\sqrt{c})x + \sqrt{c}y$, followed by the coordinate change in the target given by $\bar{Y} = Y - (a - b^2/c)X^2$ transform f into:

$$(x, y) \mapsto (x, y^2 + G(x, y), H(x, y)),$$

for some $G \in \mathfrak{m}_2^3$ and $H \in \mathfrak{m}_2^2$.

Now we use the fact that the fold $(x, y) \to (x, y^2)$ is 2-determined. This implies that there are coordinate changes in the source and the target which transform the above map germ into:

$$(x, y) \mapsto (x, y^2, K(x, y)),$$

for some $K \in \mathfrak{m}_2^2$. Finally, by the Malgrange Preparation Theorem, we split K as

$$K(x, y) = K_1(x, y^2) + yK_2(x, y^2).$$

We take the coordinate change in the target given by $\bar{Z} = Z - K_1(X, Y)$, which now transforms the map germ into

$$(x, y) \mapsto (x, y^2, yK_2(x, y^2)).$$

$\square$

Theorem 5.10 ([22]). *Any FD germ $f \in \mathscr{E}(2, 3)$ with Boardman type $\Sigma^{1,0}$ has a doodle of type "warm" (see fig. 13). In particular, two FD germs with Boardman type $\Sigma^{1,0}$ are C^0-$\mathscr{A}$-equivalent if and only if their double point curves have the same number of half-branches.*

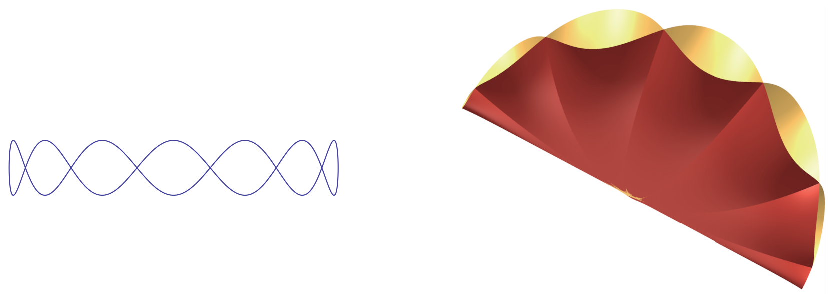


FIGURE 13. Singularity of type $\Sigma^{1,0}$ with six crossings

Proof. We can assume $f(x,y)=(x,y^2,yp(x,y^2))$. We consider $f:U\to V$ a good representative and $\epsilon>0$ a Milnor-Fukuda radius. The doodle is given by $f|_{\tilde S^1_\epsilon}:\tilde S^1_\epsilon\to S^2_\epsilon$. We have $D(f)=\{(x,y):\ p(x,y^2)=0\}$ and

$$\tilde S^1_\epsilon=\{(x,y):\ x^2+y^4+y^2p(x,y^2)^2=\epsilon^2\},$$

and both sets are symmetric with respect to the x-axis.

We choose $z_0=(\epsilon,0)$ as the base point of $\tilde S^1_\epsilon$. The crossings of the doodle are determined by $D(f)\cap\tilde S^1_\epsilon$, which gives: $z_1,\dots,z_r$ and $\overline{z}_1,\dots,\overline{z}_r$, with

$$z_i=(x_i,y_i),\quad \overline{z}_i=(x_i,-y_i),\quad -\epsilon<x_r\le\dots\le x_1<\epsilon,\quad y_i\ge 0.$$

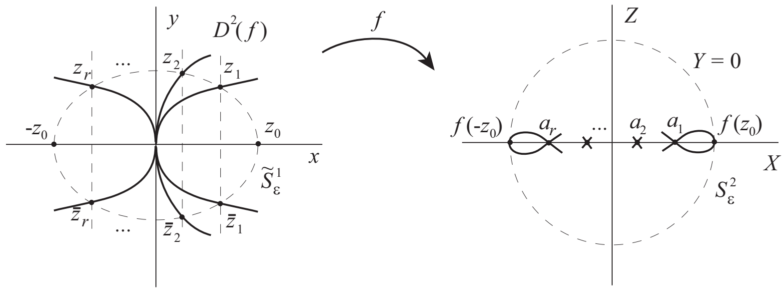


FIGURE 14. Configuration of the crossings

This implies that the Gauss word of the doodle (up to the signs) is equal to:

$$a_1a_2\dots a_ra_r\dots a_2a_1,$$

where $a_i=f(z_i)=f(\overline{z}_i)$ (see fig. 14). The doodle has the following properties:

- The doodle is contained in the hemisphere $Y\ge 0$ of S^2_ϵ and intersects the equator $Y=0$ at the base point $f(z_0)$ and its opposite $f(-z_0)$.
- The doodle is symmetric with respect to the meridian $Z=0$.
- The doodle intersects the meridian $Z=0$ only at the double points $a_1,\dots,a_r$, together with $f(z_0)$ and $f(-z_0)$. Moreover, they present the following relative position on the meridian:

$$f(-z_0)<a_r<\dots<a_1<f(z_0).$$

The only possible doodles which satisfy these properties are those of type "warm", with Gauss word:

$$a_1a_2^{-1}\dots a_r^{\pm1}a_r^{\mp1}\dots a_2a_1^{-1}.$$

□

We remark that any doodle of type "warm" with r is crossings is realizable as the link of a FD $f\in\mathscr{E}(2,3)$. In fact, we consider:

$$f(x,y)=(x,y^2,\Im((x+iy)^{r+1})),$$

where $\Im(z)$ is imaginary part of $z\in\mathbb{C}$. Then, we have

$$p(x,y^2)=\Im((x+iy)^{r+1})/y=\prod_{k=1}^{r}(-\sin(\frac{k\pi}{r+1})x+\cos(\frac{k\pi}{r+1})y),$$

hence $D(f) = \{(x, y) :\ p(x, y^2) = 0\}$ has exactly $2r$ half-branches.

6. Reeb graphs

In this section we consider the topological classification of FD germs $f \in \mathscr{E}(3, 2)$ with isolated zeros, that is, $f^{-1}(0) = \{0\}$. By Theorem 4.1, the link is a stable mapping $\gamma : S^2 \to S^1$, that is, it has only Morse singularities with distinct critical values. The combinatorial model to describe this type of mappings is given by the Reeb graph. The Reeb graph was introduced by Reeb in [37] and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$ (see [1, 39]). In this section we extend the concept of Reeb graph for stable maps from S^2 to S^1. All the results of this section appear in the paper [2].

The following result is probably well known for fibre bundles (that is, locally trivial fibrations), but we include here a elementary proof for the sake of completeness.

Lemma 6.1. *Let $p : E \to B$ be a fibre bundle with fibre F, where B, E, F are all finite CW-complexes. Then,*

$$\chi(E) = \chi(B)\chi(F).$$

Proof. After subdivision, we can choose a finite covering $\{U_i\}_{i=1}^k$ of B which trivializes the fibre bundle and such that each U_i is a subcomplex of B. For each i, there exists a homeomorphism $\varphi_i : p^{-1}(U_i) \to U_i \times F$ such that $\pi_1 \circ \varphi_i = p$, where π_1 is the projection onto the first factor. In particular, we have that $p^{-1}(A)$ is homeomorphic to $A \times F$, for any subset $A \subseteq U_i$ and $i = 1, \dots, k$.

Let $B_i = \cup_{j=1}^i U_j$, then we see that $\chi(p^{-1}(B_i)) = \chi(B_i)\chi(F)$ by induction on i. In fact, this is true for $i = 1$ and if we assume it for i, then

$$\begin{aligned}
\chi(p^{-1}(B_{i+1})) &= \chi(p^{-1}(B_i)) + \chi(p^{-1}(U_{i+1})) - \chi(p^{-1}(B_i \cap U_{i+1})) \\
&= \chi(B_i)\chi(F) + \chi(U_{i+1})\chi(F) - \chi(B_i \cap U_{i+1})\chi(F) \\
&= (\chi(B_i) + \chi(U_{i+1}) - \chi(B_i \cap U_{i+1}))\chi(F) \\
&= \chi(B_{i+1})\chi(F).
\end{aligned}$$

□

Proposition 6.2. *Let $\gamma : S^2 \to S^1$ be a stable map. Then γ is not a regular map.*

Proof. Suppose γ is a regular map, then $\gamma(S^2) \subset S^1$ would be an open set. Since $\gamma(S^2)$ is also closed, we get $\gamma(S^2) = S^1$ and hence, γ is surjective. By Ehresmann's fibration theorem [7, page 31], f is a smooth fibre bundle. In particular, if F is the fiber, we have by Lemma 6.1 that

$$2 = \chi(S^2) = \chi(S^1)\chi(F) = 0,$$

which is an absurd. □

Given a continuous map $f : X \to Y$ between topological spaces, we consider the following equivalence relation on X: $x \sim y$ if $f(x) = f(y)$ and x and y are in the same connected component of $f^{-1}(f(x))$.

Proposition 6.3. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the quotient space $S^2/\sim$ admits the structure of a connected graph in the following way:*

(1) *the vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
(2) *each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof. Since γ is stable we have a finite number of critical values $v_1, \dots, v_r$ and for each $i = 1, \dots, r$, $\gamma^{-1}(v_i)$ has a finite number of connected components. Then,

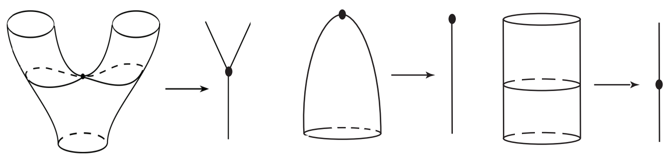
$$\gamma | S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}) : S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}) \to S^1 - \{v_1, \dots, v_r\}$$

is regular, and the induced map

$$\tilde{\gamma} : (S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim \to S^1 - \{v_1, \dots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \dots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim$ is also homeomorphic to an open interval. □

Each vertex of the graph can be of three types, depending on if the connected component has a maximum/minimum critical point, a saddle point or just regular points. Then, the possible incidence rules of edges and vertices are given in fig. 15.



a b c

Figure 15. Incidence rules for the three types of vertices

Let $v_1, \dots, v_r \in S^1$ be the critical values of γ. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \dots < v_r$ and we label each vertex with the index $i \in \{1, \dots, r\}$, if it corresponds to the critical value v_i.

Definition 6.4. The graph given by $S^2/\sim$ together with the the labels of the vertices, as previously defined, is said to be the *generalized Reeb graph* associated to $\gamma : S^2 \to S^1$ (see fig. 16).

For simplicity, from now on we will just call Reeb graph to the generalized Reeb graph, unless otherwise specified.

Proposition 6.5. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the Reeb graph of γ is a tree.*

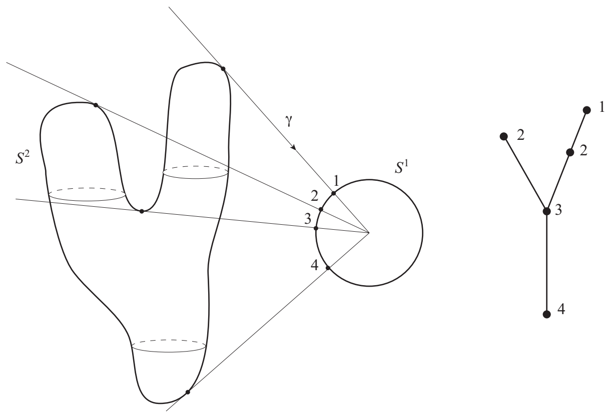


FIGURE 16. Example of Reeb graph of a stable map $\gamma : S^2 \to S^1$

Proof. Let Γ be the Reeb graph of γ. Since Γ is connected, in order to show that Γ is a tree, we only need to prove that its Euler characteristic is $\chi(\Gamma) = 1$. We have that $\chi(\Gamma) = V - E$, where V, E are the number of vertices and edges of Γ, respectively.

On one hand, $V = M + S + I$ where M, S, I are the numbers of vertices of each type: maximum/minimum, saddle or regular, respectively. Note that $V \neq 0$ by Proposition 6.2.

On the other hand, by Euler's formula $E = \frac{1}{2}\sum \deg(v_i)$ where v_i are the vertices of Γ and $\deg(v_i)$ is the degree of v_i, that is, the number of edges adjacent to v_i. Since γ is stable, the degree of each vertex of maximum/minimum type is 1, while of regular type is 2 and of saddle type is 3 (see fig. 15). Hence,

$$\chi(\Gamma) = V - E = M + S + I - \frac{1}{2}(M + 2I + 3S) = \frac{M - S}{2} = 1,$$

where the last equality follows from the Morse formula: $M - S = \chi(S^2) = 2$. $\square$

Remark 6.6. The classical Reeb graph is defined in the same way, but the vertices are just the connected components of level curves $\gamma^{-1}(v)$ which contain a critical point. Hence, our generalized Reeb graph contains some extra vertices corresponding to the regular connected components of $\gamma^{-1}(v)$, where v is a critical value. Of course the classical Reeb graph can be obtained from the generalized one just by eliminating the extra vertices and joining the two adjacent edges. But in general, the generalized Reeb graph provides more information.

We present in fig. 17 two examples of stable maps $\gamma_1, \gamma_2 : S^2 \to S^1$ with their respective generalized Reeb graphs. Both examples share the same classical Reeb graph, but the generalized Reeb graphs are different. The example on the left hand side is a non-surjective map, whilst the map on the right hand side is surjective, therefore the maps are not topologically equivalent. This shows that the classical Reeb graph is not sufficient to distinguish between these two examples.

Notice that if $\gamma : S^2 \to S^1$ is not surjective, then γ may be regarded as a Morse function from S^2 to $\mathbb{R}$ (via stereographic projection). In this case, the generalized Reeb graph can be deduced from the classical one just by adding the extra vertices each time that one passes through a critical value.

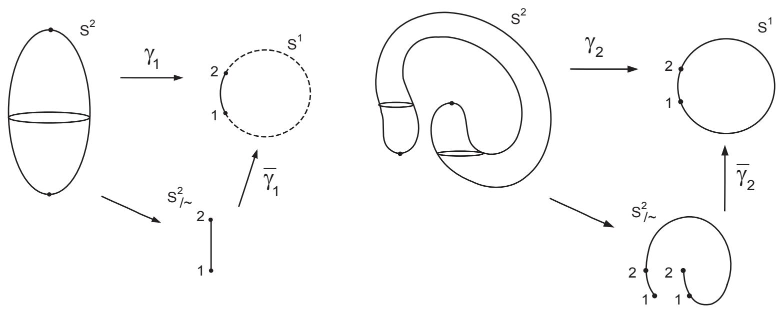


FIGURE 17. Two non-equivalent stable maps with the same classical Reeb graph

It is obvious that labeling of vertices of the Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or a reversion of the labeling in the Reeb graph. This leads us to the following definition of equivalent Reeb graphs.

Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Let Γ_γ and Γ_δ be their respective Reeb graphs. Consider the induced quotient maps $\bar{\gamma} : \Gamma_\gamma \to S^1_\gamma$ and $\bar{\delta} : \Gamma_\delta \to S^1_\delta$, where S^1_γ, S^1_δ is S^1 with the graph structure whose vertices are the critical values of γ, δ respectively (as illustrated in fig. 17).

Definition 6.7. We say that Γ_γ *is equivalent to* Γ_δ and we denote it by $\Gamma_\gamma \sim \Gamma_\delta$, if there exist graph isomorphisms $j : \Gamma_\gamma \to \Gamma_\delta$ and $l : S^1_\gamma \to S^1_\delta$, such that the following diagram is commutative:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{\bar{\gamma}|V_\gamma} & \Delta_\gamma \\ {\scriptstyle j|_{V_\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle l|_{\Delta_\gamma}} \\ V_\delta & \xrightarrow{\bar{\delta}|V_\delta} & \Delta_\delta \end{array}$$

where $V_\gamma = \{\text{vertices of } \Gamma_\gamma\}$, $V_\delta = \{\text{vertices of } \Gamma_\delta\}$ and Δ_γ and Δ_δ are their respective discriminant sets.

Theorem 6.8. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. If γ and δ are C^0-$\mathscr{A}$-equivalent then their respective Reeb graphs are equivalent.*

Proof. Since γ and δ are topologically equivalent there exist homeomorphisms $h : S^2 \to S^2$ and $k : S^1 \to S^1$ such that $k \circ \gamma \circ h = \delta$. Then h maps critical points into critical points and k maps critical values into critical values. Hence h induces a graph isomorphism from Γ_γ to Γ_δ and k induces a graph isomorphism from S^1_γ to S^1_δ which gives the equivalence between the Reeb graphs. □

The above theorem allows us to extend the definition of Reeb graph for C^0-stable maps between topological spheres.

Definition 6.9. Let $\gamma : M \to P$ be a continuous map, where M is homeomorphic to S^2 and P is homeomorphic to S^1. We say that γ is C^0*-stable* if there exist a C^∞-stable map $\delta : S^2 \to S^1$ and homeomorphisms $k : M \to S^2$, $h : P \to S^1$ such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & P \\ k\downarrow & & \downarrow h \\ S^2 & \xrightarrow{\delta} & S^1 \end{array}$$

We say that $y \in P$ is a *critical value* of γ if $h(y)$ is a critical value of δ. Moreover, $M/\sim$ has a graph structure induced by the Reeb graph of δ. We call this graph the *Reeb graph* of γ and denote it by Γ_γ. The notion of equivalence of graphs given in Definition 6.7 can be also extended for C^0-stable maps in the obvious way. By Theorem 6.8, the Reeb graph Γ_γ is well defined up to equivalence of graphs.

The main result is the following theorem which says that the Reeb graph is a complete invariant for $\mathscr{A}$-equivalence of stable maps from S^2 to S^1. The idea of the proof is that we can "inflate" the Reeb graph and then recover the surface together with the stable map. Near each vertex, we have a Morse singularity and the local normal form is given in fig. 14. Along the edges, the map is regular, so we have pieces of "tubes" which connect the singularities. The detailed proof, although intuitive, is rather technical and in fact is an adaptation of the proof of [14, Theorem 4.1]. All the details can be found in [2, Theorem 3.8].

Theorem 6.10. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps such that $\Gamma_\gamma \sim \Gamma_\delta$. Then γ is $\mathscr{A}$-equivalent to δ.*

As we said before, the two theorems 6.8 and 6.10 together give that the Reeb graph is a complete topological invariant for stable maps from S^2 to S^1. In fact, we have a little bit more, as we can see in the following corollary.

Corollary 6.11. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Then the following statements are equivalent:*

(1) *γ, δ are $\mathscr{A}$-equivalent,*
(2) *γ, δ are C^0-$\mathscr{A}$-equivalent,*
(3) *$\Gamma_\gamma \sim \Gamma_\delta$.*

In the last part of this section, we consider the Reeb graph of the link of a finitely determined map germ with isolated zeros.

Definition 6.12. Given a FD germ $f \in \mathscr{E}(3, 2)$ with $f^{-1}(0) = \{0\}$, we define the *Reeb graph* of f as the Reeb graph of the link of f.

It follows from Theorem 6.10 and Corollary 4.6 that if two FD germs have equivalent Reeb graphs, then they are C^0-$\mathscr{A}$-equivalent. Again in this case we can show the converse. But we need to see how is the structure of a FD germ in this case. The first step is to describe the stable singularities. The characterization of stable singularities of maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ is well known (cf. [12]) and it is given by:

Theorem 6.13. *Let $f : (\mathbb{R}^3, S) \to (\mathbb{R}^2, 0)$ be a C^∞ multi-germ germ such that f is singular at each point of S. Then, f is stable if only if $|S| \le 2$ and f is $\mathscr{A}$-equivalent to one of the following normal forms:*

(1) *For $|S| = 1$:*
   - $(x, y^2 + z^2)$, *called definite fold D;*
   - $(x, y^2 - z^2)$, *called indefinite fold I;*
   - $(x, y^3 + xy + z^2)$, *called cusp.*

(2) *For $|S| = 2$:*
   - $(x_1, y_1^2 + z_1^2), (y_2^2 + z_2^2, x_2)$, *called double-fold D&D;*
   - $(x_1, y_1^2 + z_1^2), (y_2^2 - z_2^2, x_2)$, *called double-fold D&I;*
   - $(x_1, y_1^2 - z_1^2), (y_2^2 - z_2^2, x_2)$, *called double-fold I&I.*

Proof. We follow the same arguments as in Example 2.12 and Theorem 5.5. We first consider the mono-germ case $|S| = 1$. If f is a fold (either definite or indefinite), then

$$\begin{aligned} T\mathscr{K}_e f &= \mathscr{E}_3 \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2y \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 2z \end{pmatrix} \right\} + \langle x, y^2 \pm z^2 \rangle \mathscr{E}_3^2 \\ &= \mathscr{E}_3 \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right\}. \end{aligned}$$

Thus $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the class of $(0,1)$ and the map $\overline{\omega} f$ is obviously surjective, so f is stable (see Lemma 2.9). In the case of the cusp, we have:

$$\begin{aligned} T\mathscr{K}_e f &= \mathscr{E}_3 \left\{ \begin{pmatrix} 1 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^2 + x \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 2z \end{pmatrix} \right\} + \langle x, y^3 + xy + z^2 \rangle \mathscr{E}_3^2 \\ &= \mathscr{E}_3 \left\{ \begin{pmatrix} 1 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right\}. \end{aligned}$$

Now, $\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{R}$ by the classes of $\{(1,0),(0,1)\}$. Again $\overline{\omega} f$ is surjective and hence, f is stable.

Assume now that $f \in \mathscr{E}(3,2)$ is stable. If f has rank 0, then $T\mathscr{K}_e f \subset \mathfrak{m}_3 \theta(f)$. Since $\theta(f)/\mathfrak{m}_2\theta(f)$ has dimension 2, we must have necessarily that $T\mathscr{K}_e f = \mathfrak{m}_2 \theta(f)$. Moreover, $(f^*\mathfrak{m}_2) \subset \mathfrak{m}_3^2 \theta(f)$, hence the classes of $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ should generate $\mathfrak{m}_3\theta(f)/\mathfrak{m}_3^2\theta(f)$ over $\mathbb{R}$. But this is not possible, since this space has dimension 6.

Thus, if f is stable, it must have rank 1 and after a coordinate change in the source, we can assume that $f(x,y,z) = (x, g(x,y,z))$, for some function $g \in \mathfrak{m}_3^2$. In other words, we see f as an unfolding of $g_0(y,z) = g(0,y,z)$. In particular, we have:

$$\frac{\theta(f)}{T\mathscr{K}_e(f)} \cong \frac{\theta(g_0)}{T\mathscr{K}_e(g_0)} \cong \frac{\mathscr{E}_2}{\left\langle \frac{\partial g_0}{\partial y}, \frac{\partial g_0}{\partial z}, g_0 \right\rangle}.$$

Let $I = \left\langle \frac{\partial g_0}{\partial y}, \frac{\partial g_0}{\partial z}, g_0 \right\rangle$. If $g_0 \in \mathfrak{m}_2^3$, then $I \subset \mathfrak{m}_2^2$ and thus $\dim_{\mathbb{R}}(\mathscr{E}_2/I) \ge 3$, which is not possible by the surjectivity of $\overline{\omega} f$. Hence, the Hessian matrix of g_0 at the origin must have rank ≥ 1. By the splitting lemma, g_0 is $\mathscr{A}$-equivalent to $y^{k+1} \pm z^2$, for some $k \ge 1$. This implies $\dim_{\mathbb{R}}(\mathscr{E}_2/I) = k$, hence we must have necessarily $k \le 2$. If $k = 1$, then f is a fold (either definite or indefinite) and if $k = 2$, then f is a cusp, by Theorem 2.11.

We consider now multi-germs $f : (\mathbb{R}^3, S) \to (\mathbb{R}^2, y)$, with $S \subset \mathbb{R}^3$ a finite set. If one of the points $x_i \in S$ is a cusp, then the analytic stratum is only the point $\{x_i\}$. Thus, the regular intersection condition of Theorem 2.15 implies that $S = \{x_i\}$. Otherwise, if all the points of S are folds, the analytic stratum at each point is a line. The regular intersection condition now implies that $|S| \leq 2$ and that the two lines are transverse in the plane in the case $|S| = 2$. This implies that f is a double-fold. □

Note that the 0-stable types are the cusps and the double-folds. Hence if $f \in \mathscr{E}(3, 2)$ is FD, then there exists a good representative $f : U \to V$ such that

(1) $S(f) \cap f^{-1}(0) = \{0\}$,
(2) the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ has only definite and indefinite simple fold singularities.

We have that $S(f)$ and the discriminant $\Delta(f) = f(S(f))$ are curves which are regular outside the origin. After shrinking U, V if necessary, we can assume that $S(f), \Delta(f)$ are made of a finite number of arcs joining the origin with the boundary of U, V, called *half-branches*. Moreover, the restriction $f : S(f) \setminus \{0\} \to \Delta(f) \setminus \{0\}$ is a diffeomorphism. Each half-branch of $\Delta(f)$ corresponds to a critical value of the link of f, which is of type max/min if we are in a half-branch of type definite fold and of type saddle if we are in a half-branch of type indefinite fold. Another important set is

$$X(f) = \overline{f^{-1}(\Delta(f)) \setminus S(f)}.$$

The set $X(f)$ is a regular surface outside the origin and will also assume that the connected components of $X(f) \setminus \{0\}$ are cylinders going from the origin to the boundary of U. Each half-branch of $S(f)$ corresponds to a vertex of the Reeb graph of type max/min if we are in a half-branch of type definite fold and of type saddle if we are in a half-branch of type indefinite fold. Each connected component of $X(f) \setminus \{0\}$ corresponds to a regular vertex of the Reeb graph.

Theorem 6.14. *Let $f, g \in \mathscr{E}(3, 2)$ be FD germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. If f and g are C^0-$\mathscr{A}$-equivalent then their Reeb graphs are equivalent.*

Proof. By hypothesis, there exist two homeomorphisms germs h, k such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^3, 0) & \xrightarrow{f} & (\mathbb{R}^2, 0) \\ h \downarrow & & \downarrow k \\ (\mathbb{R}^3, 0) & \xrightarrow{g} & (\mathbb{R}^2, 0) \end{array} \tag{1}$$

We take representatives of f, g, h and k and for any small enough $\epsilon > 0$, the next diagram is also commutative:

$$\begin{array}{ccc} \tilde{S}^2_\epsilon & \xrightarrow{\gamma_f} & S^1_\epsilon \\ h \downarrow & & \downarrow k \\ M_\epsilon & \xrightarrow{g|_{M_\epsilon}} & P_\epsilon \end{array} \tag{2}$$

where $M_\epsilon = h(\tilde{S}^2_\epsilon)$ and $P_\epsilon = k(S^1_\epsilon)$.

From the commutativity of diagram (2) follows that $g|M_\epsilon$ is C^0-stable. Choose $\epsilon_0, \epsilon_1 > 0$ such that $\gamma_f : \tilde{S}^2_{\epsilon_0} \to S^1_{\epsilon_0}$ and $\gamma_g : \tilde{S}^2_{\epsilon_1} \to S^1_{\epsilon_1}$ are the links of f and g, respectively, and $S^1_{\epsilon_1} \subset k(D^2_{\epsilon_0})$. By Definition 6.9, let $\Gamma_{g|M_{\epsilon_0}}$ be the Reeb graph associated to $g|M_{\epsilon_0}$. Then, we can conclude that $\Gamma_{g|M_{\epsilon_0}}$ is equivalent to Γ_{γ_f}, where Γ_{γ_f} is the Reeb graph of γ_f.

Consider $A_1, \dots, A_n$ the half branches of the discriminant $\Delta(g)$ ordered in the anti-clockwise orientation. By the cone structure of f (see Theorem 4.1), each half branch A_i intersects P_{ϵ_0} in a unique point v_i so that $v_1, \dots, v_n$ are the critical points of $g|M_{\epsilon_0}$. Analogously, each A_i intersects $S^1_{\epsilon_1}$ in a unique point w_i, where now $w_1, \dots, w_n$ are the critical points of γ_g. We have a graph isomorphism $l : P_{\epsilon_0} \to S^1_{\epsilon_1}$ given by $l(v_i) = w_i$, $\forall i = 1, \dots, n$.

Let $C_1, \dots, C_r$ be the connected components of

$$g^{-1}(\Delta(g)) \setminus \{0\} = \cup_{i=1}^n g^{-1}(A_i).$$

Again by the cone structure of f, each connected component C_j intersects M_{ϵ_0} in a unique connected component V_j of some $g^{-1}(v_i)$, so that $V_1, \dots, V_r$ are the vertices of $\Gamma_{g|M_{\epsilon_0}}$. Finally, each C_j intersects $\tilde{S}^2_{\epsilon_1}$ in a unique connected component W_j of $g^{-1}(w_i)$, in such a way that $W_1, \dots, W_r$ are now the vertices of Γ_{γ_g}. We have a bijection φ defined by $\varphi(V_j) = W_j$, $\forall j = 1, \dots, r$. In order to have a graph isomorphism between $\Gamma_{g|M_{\epsilon_0}}$ and Γ_{γ_g} we need to show that φ is edge preserving.

Consider $U = k(D^2_{\epsilon_0}) \setminus (\Delta(g) \cup B^2_{\epsilon_1})$, and let Y_i be one of its connected components limited by two consecutive half branches A_i and A_{i+1}. We denote by α_i and β_i the arcs of $S^1_{\epsilon_1}$ and P_{ϵ_0} respectively, which bound Y_i, $\forall i = 1, \dots, n$ (see fig. 18). The connected components of $g^{-1}(\alpha_i)$ and $g^{-1}(\beta_i)$ give all the edges of the graphs Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$, respectively.

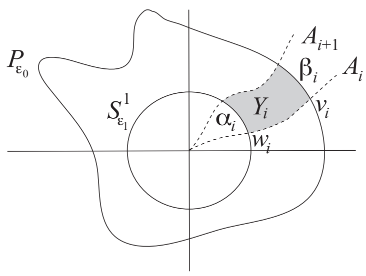


FIGURE 18

Take X any connected component of $f^{-1}(Y_i)$, for some $1 \le i \le n$. Since $g|X : X \to Y_i$ is regular, the induced map $\tilde{g} : X/\sim \to Y_i$ is a local homeomorphism and hence, a covering space. But Y_i is simply connected, so $\tilde{g}$ is in fact a homeomorphism. We deduce that the boundary of $X/\sim$ has two components: one is an edge of Γ_{γ_g} given by the quotient of $X \cap g^{-1}(\alpha_i)$ and the other is an edge of $\Gamma_{g|M_{\epsilon_0}}$ given by the quotient of $X \cap g^{-1}(\beta_i)$.

Notice that all the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$ can be obtained in this way, hence we have a bijection between the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$ which is compatible with the above bijection φ defined between the vertices. $\square$

Again, Theorem 6.14 together with Corollary 4.6 and Theorem 6.10 show that the Reeb graph is a complete topological invariant for map germs from with isolated zeros.

Corollary 6.15. *Let $f, g \in \mathscr{E}(3,2)$ be FD germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. Then the following statements are equivalent:*

(1) *f, g are C^0-$\mathscr{A}$-equivalent,*
(2) *the Reeb graphs of f, g are equivalent,*
(3) *the links of f, g are C^0-$\mathscr{A}$-equivalent.*

As we did in Section 5, in the last part of this section, we will describe the topology of FD germs $f \in \mathscr{E}(3,2)$ with Boardman type $\Sigma^{2,1}$. These germs constitute the simplest non trivial class of singular germs. The Boardman type Σ^2 means that f has corank 1 and the next result gives a restriction on the link for this class of germs.

Lemma 6.16. *Let $f \in \mathscr{E}(3,2)$ be a corank 1 FD germ given by $f(x,y,z) = (x, h_x(y,z))$. Then $h_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is FD.*

Proof. Since f is FD, we can assume it is polynomial. Then its complexification $f_{\mathbb{C}}$ is also FD and by the Mather-Gaffney criterion $S(f_{\mathbb{C}}) \cap f_{\mathbb{C}}^{-1}(0) = \{0\}$ (see 3.4). This implies that $S((h_0)_{\mathbb{C}}) \cap (h_0)_{\mathbb{C}}^{-1}(0) = \{0\}$ and hence h_0 is FD for the contact group $\mathscr{K}$. But for function germs, it is well-known that the FD with respect the contact group $\mathscr{K}$ is equivalent to the FD with respect to the group $\mathscr{A}$ (see again [42, Proposition 2.3]). □

Theorem 6.17. *Let $f \in \mathscr{E}(3,2)$ be a corank 1 FD germ with $f^{-1}(0) = \{0\}$. Then the link of f is not surjective.*

Proof. Consider f written by $f(x,y,z) = (x, h_x(y,z))$, where h_0 is also FD and $h_0^{-1}(0) = \{0\}$. By Theorem 4.1, $h_0^{-1}(S_\epsilon^0)$ is diffeomorphic to S^1, for small enough $\epsilon > 0$.

Suppose that associated link of f is surjective. Then $(0, \epsilon)$ and $(0, -\epsilon)$ belong to image of the map $\gamma_f : f^{-1}(S_\epsilon^1) \to S_\epsilon^1$. But

$$\gamma_f^{-1}(\{(0,\epsilon), (0,-\epsilon)\}) = f^{-1}(\{(0,\epsilon), (0,-\epsilon)\}) \simeq h_0^{-1}(\{\epsilon, -\epsilon\}) \simeq S^1,$$

where $\simeq$ indicates homeomorphism of sets. This gives a contradiction because S^1 is connected, $\{(0,\epsilon), (0,-\epsilon)\}$ is not connected and γ_f is a continuous map. □

Remark 6.18. (1) It follows from Theorem 6.17 that the stable map $\gamma : S^2 \to S^1$ presented in the right hand side of fig. 17 cannot be realized as the link of a corank 1 FD map germ $f \in \mathscr{E}(3,2)$. Up to this moment, we do not know if in fact, this stable map can be realized or not as the link of a corank 2 map germ.

(2) Another consequence of Theorem 6.17 is that if f has corank 1 and $f^{-1}(0) = \{0\}$, then the generalized Reeb graph can obtained from the classical one, since the link is not surjective (see Remark 6.6). From now on in this section, the Reeb graph will be referred to the classical version, unless otherwise specified.

Any corank 1 germ $f \in \mathscr{E}(3,2)$ may have Boardman type $\Sigma^{2,0}$ or $\Sigma^{2,1}$, $\Sigma^{2,2}$. It is easy to see that if f has type $\Sigma^{2,0}$, then it is $\mathscr{A}$-equivalent to the definite or indefinite fold $(x,y,z) \mapsto (x, y^2 \pm z^2)$, so we do not need to consider this case. From now on, we restrict ourselves to germs of type $\Sigma^{2,1}$.

Lemma 6.19. *Any FD germ $f \in \mathscr{E}(3,2)$ of Boardman type $\Sigma^{2,1}$ with $f^{-1}(0) = \{0\}$ can be written, up to $\mathscr{A}$-equivalence, as*

$$f(x,y,z) = (x, y^k + a_{k-2}(x)y^{k-2} + \cdots + a_1(x)y + z^2), \tag{3}$$

for some $k \geq 4$ even and functions $a_1, \ldots, a_{k-2} \in \mathscr{E}_1$.

Proof. Consider f written by $f(x,y,z) = (x, h_x(y,z))$, where h_0 is also FD and $h_0^{-1}(0) = \{0\}$. The fact f has type $\Sigma^{2,1}$ implies that the Hessian of h_0 has rank 1, hence up to $\mathscr{A}$-equivalence, h_0 is given by $h_0(y,z) = y^k + z^2$, for some $k \geq 4$ even. The mini-versal deformation of h_0 is

$$H(a_1, \ldots, a_{k-2}, y, z) = y^k + a_{k-2}y^{k-2} + \cdots + a_1 y + z^2.$$

Then, there exist functions $a_1, \ldots, a_{k-2} \in \mathscr{E}_1$ such that

$$f(x,y,z) = (x, H(a_1(x), \ldots, a_{k-2}(x), y, z)).$$

□

Definition 6.20. We say that a FD germ $f \in \mathscr{E}(3,2)$ of Boardman type $\Sigma^{2,1}$ with $f^{-1}(0) = \{0\}$ has *multiplicity* k, if it can be written, up to $\mathscr{A}$-equivalence as in (3).

Let $f \in \mathscr{E}(3,2)$ be FD germ of Boardman type $\Sigma^{2,1}$ with $f^{-1}(0) = \{0\}$ and multiplicity k given as in (3). We write, for simplicity,

$$h_x(y) = y^k + a_{k-2}(x)y^{k-2} + \cdots + a_1(x)y.$$

We fix a good representative $f : U \to V$ and take $\epsilon > 0$ such that $(-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \subset U$. The singular points of f are points $(x,y,0)$ such that $h_x'(y) = 0$. The fact that f has fold type outside the origin implies if $x \neq 0$, then $h_x''(y) \neq 0$ at the singular points. Moreover, f has a definite fold if $h_x''(y) > 0$ and an indefinite fold if $h_x''(y) < 0$. Moreover, all the critical values of have to be distinct.

We deduce that x with $0 < |x| < \epsilon$, the function $h_x : (-\epsilon,\epsilon) \to \mathbb{R}$ is a Morse with distinct critical values. In particular, all the functions h_x with $0 < x < \epsilon$ are $\mathscr{A}$-equivalent and all the functions h_x with $-\epsilon < x < 0$ are also $\mathscr{A}$-equivalent. In both we have a Morsification of x^k and the relative position of the critical values in both functions determine the Reeb graph of f.

Since k is even, h_x will have an odd number of critical points $y_1, \ldots, y_r$ with $r \leq k-1$. The points $y_1, y_3, \ldots, y_r$ are the local minima and the points $y_2, y_4, \ldots, y_{r-1}$ are the local maxima of h_x. If the critical values are $v_1 < \cdots < v_r$, then can associate with h_x a permutation $\sigma \in \Sigma_r$ such that $h_x(y_i) = v_{\sigma(i)}$. We denote by σ^+, σ^- the two permutations of h_x for $x > 0$ and $x < 0$ respectively. Then, the pair (σ^+, σ^-) determines the Reeb graph of f.

Exemple 6.21. Let $f \in \mathscr{E}(3,2)$ be FD germ of Boardman type $\Sigma^{2,1}$ with $f^{-1}(0) = \{0\}$ and multiplicity 4. After change of coordinates in the source and target, we can assume f is given by

$$f(x,y,z) = (x, y^4 + a(x)y^2 + b(x)y + z^2).$$

Notice that the bifurcation set $\mathcal{B}$ of the versal unfolding of h_0 in this case is given in the (a,b)-plane by by $b(-4a^3 - 27b^2) = 0$ (see fig. 19), which permits us to choose appropriate functions $a(x)$ and $b(x)$ such that we can obtain all types of possible configurations.

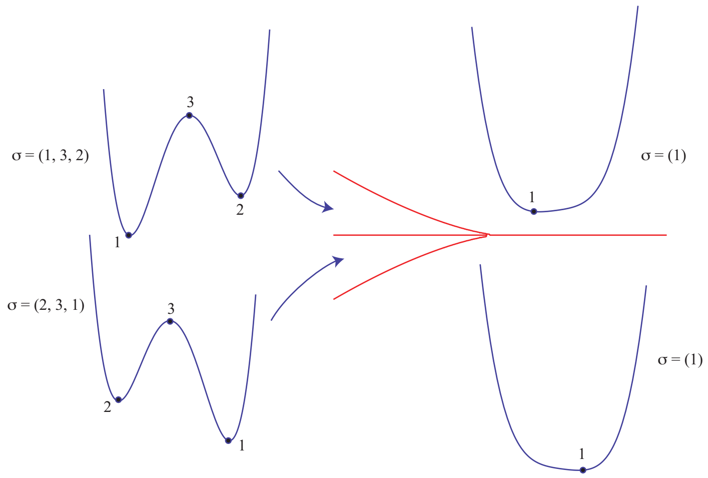


FIGURE 19. Morsifications of y^4

Then, there are three possibilities for the Reeb graph of the link of f, according to the number of saddles:

- 0 saddle, in this case $(\sigma^+, \sigma^-) = ((1),(1))$, then f is topologically equivalent to $(x, y^4 + x^2y + z^2)$ (see fig. 20);
- 1 saddle, this corresponds to $(\sigma^+, \sigma^-) = ((1),(1,3,2))$, then f is topologically equivalent to $(x, y^4 + xy^2 + 3x^5y + z^2)$ (see fig. 21);
- 2 saddles, this happens if $(\sigma^+, \sigma^-) = ((1,3,2),(1,3,2))$ and f is topologically equivalent to $(x, y^4 - x^2y^2 + x^5y + z^2)$. (see fig. 22).

We remark that the configuration $((1,3,2),(2,3,1))$ is topologically equivalent to $((1,3,2),(2,3,1))$ since the corresponding Reeb graphs are equivalent.

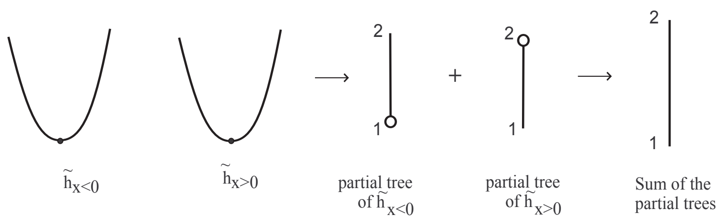


FIGURE 20. Reeb graph with no saddles

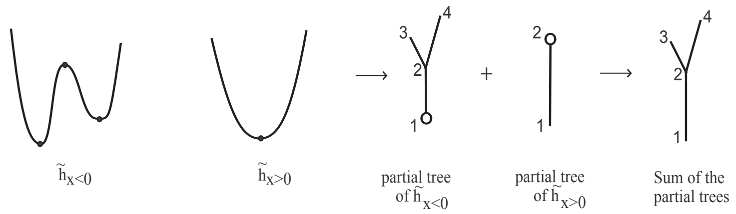


Figure 21. Reeb graph with one saddle

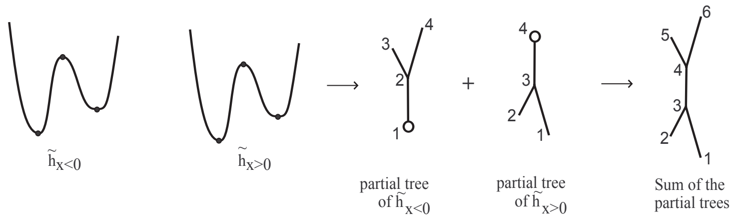


Figure 22. Reeb graph with two saddles

7. The cone structure theorem for map germs with non isolated zeros

The case of a FD germ $f \in \mathscr{E}(n,p)$ with $f^{-1}(0) \neq \{0\}$ is much more complicated than the case with $f^{-1}(0) = \{0\}$. Fukuda gave in [9] an analogous theorem to Theorem 4.1, which in our notation can be stated as follows(see [9, Theorem 1']).

Theorem 7.1. *Let $f : U \to V$ a good representative of a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II, DST and such that $f^{-1}(0) \neq \{0\}$. Then, there exist $\epsilon_0 > 0$ and a strictly increasing smooth function $\delta : [0, \epsilon_0] \to [0, +\infty)$ with $\delta(0) = 0$ such that for any ϵ, δ with $0 < \epsilon \leq \epsilon_0$ and $0 < \delta \leq \delta(\epsilon)$, the following properties hold:*

(1) *$f^{-1}(0) \cap S^{n-1}_\epsilon$ is a smooth submanifold of dimension $n-p-1$, whose diffeomorphic type is independent of ϵ.*
(2) *$N_{\epsilon,\delta} := D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ is a smooth submanifold with boundary of dimension $n-1$, whose diffeomorphic type is independent of ϵ, δ.*
(3) *The restriction $f|_{N_{\epsilon,\delta}} : N_{\epsilon,\delta} \to S^{p-1}_\delta$ is a stable mapping, whose $\mathscr{A}$-class is independent of ϵ, δ.*

The proof of this theorem can be done by using similar arguments to those of the proof of Theorem 4.1 for the case $f^{-1}(0) = \{0\}$. Of course, we can define the link of f as being the stable mapping $f|_{N_{\epsilon,\delta}} : N_{\epsilon,\delta} \to S^{p-1}_\delta$. The main problem now is that f is not C^0-$\mathscr{A}$-equivalent to the cone of $f|_{N_{\epsilon,\delta}}$ in the usual sense. In fact, since $N_{\epsilon,\delta}$ is not a sphere, its cone is not a disk. So, we need to introduce a generalized version of the cone in order to solve this. The following construction is given in [5]. We recall that if X, Y are topological spaces and $f : A \to Y$ is a continuous map on $A \subset X$, then the

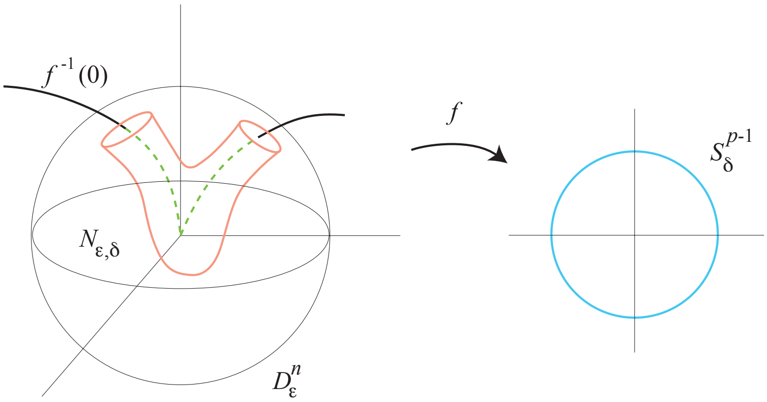


FIGURE 23. The map $f|N_{\epsilon,\delta}$.

attachment is defined as

$$X \cup_f Y = \frac{X \sqcup Y}{x \sim f(x) : \ \forall x \in A},$$

where $\sqcup$ means disjoint union and $\sim$ indicates that all points of A are identified with its images.

Definition 7.2. A *link diagram* is a diagram of the form

$$V \xleftarrow{\ r\ } N \xrightarrow{\ \gamma\ } S^{p-1},$$

where N is a manifold with boundary, γ is a continuous map, V is a contractible space and r is a continuous surjective map such that the attachment $(N \times I) \cup_r V$ is homeomorphic to the closed disk D^n (here we identify $N \equiv N \times \{0\} \subset N \times I$).

Definition 7.3. Given a link diagram $V \xleftarrow{\ r\ } N \xrightarrow{\ \gamma\ } S^{p-1}$, the *generalized cone of a link diagram* is the induced map

$$C(\gamma, r) : (N \times I) \cup_r V \to c(S^{p-1}),$$

defined in the obvious way (that is, $[x,t] \mapsto [\gamma(x), t]$ if $(x,t) \in N \times I$ and $[y] \mapsto [0]$ if $y \in V$).

Notice that here we are using the small letter c to the usual notion of cone and the capital letter C to indicate the generalized cone. Also note that in applying the notion of generalized cone of a link diagram for the case $V = \{0\}$, we obtain essentially the usual notion of the cone.

Definition 7.4. We say that two link diagrams

$$V_0 \xleftarrow{\ r_0\ } N_0 \xrightarrow{\ \gamma_0\ } S^{p-1}, \quad V_1 \xleftarrow{\ r_1\ } N_1 \xrightarrow{\ \gamma_1\ } S^{p-1}$$

are *$\mathscr{A}$-equivalent* (resp. *C^0-$\mathscr{A}$-equivalent*) if there are diffeomorphisms (resp. homeomorphisms) $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$.

The following lemma follows easily from the definitions.

Lemma 7.5. *If two link diagrams are C^0-$\mathscr{A}$-equivalent, then their generalized cones are C^0-$\mathscr{A}$-equivalent.*

We present now the structure cone theorem for map germs with non isolated zeros. Let $f \in \mathscr{E}(n,p)$, in order to simplify the notation, we put $f_{\epsilon,\delta} := f|_{N\epsilon,\delta} : N_{\epsilon,\delta} \to S^{p-1}_\delta$ and $V_\epsilon = f^{-1}(0) \cap D^n_\epsilon$.

Theorem 7.6. [4] *Let $f : U \to V$ a good representative of a polynomial map germ $f \in \mathscr{E}(n,p)$ with II, DST and such that $f^{-1}(0) \neq \{0\}$. For each ϵ, δ with $0 < \delta \ll \epsilon \ll 1$, there exists a continuous and surjective mapping $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$, such that:*

(1) *The link diagram*

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^{p-1}_\delta$$

is independent of ϵ, δ up to C^0-$\mathscr{A}$-equivalence.

(2) *The restriction $f|_{D^n_\epsilon \cap f^{-1}(D^p_\delta)} : D^n_\epsilon \cap f^{-1}(D^p_\delta) \to D^p_\delta$ is C^0-$\mathscr{A}$-equivalent to the generalized cone:*

$$C(f_{\epsilon,\delta}, r_{\epsilon,\delta}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S^{p-1}_\delta),$$

where $I = [0,\delta]$.

Here we give a sketch of the proof of Theorem 7.6, full details of the proof can be found in [4, Theorem 4.4]. Let $(\mathcal{A}, \mathcal{B})$ be the stratification by stable types of f, which is a Thom stratification of f. We choose $\epsilon_0 > 0$ and $0 < \delta_0 \ll \epsilon_0 \ll 1$ small enough and denote by $B^n_{\epsilon_0}, B^p_{\delta_0}$ the interiors of $D^n_{\epsilon_0}, D^p_{\delta_0}$ respectively. Then $D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ is a manifold with boundary. We consider the mappings

$$D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}) \xrightarrow{f} B^p_{\delta_0} \xrightarrow{\rho} [0, \delta_0),$$

where $\rho(y) = \|y\|^2$. Both are proper and we have that the restriction of $(\mathcal{A}, \mathcal{B})$ is a Thom stratification of f and $(\mathcal{B}, \mathcal{C})$ is a Thom stratification of ρ, where $\mathcal{C} = \{(0,\delta_0), \{0\}\}$. We take stratified vector fields X, Y, T on $D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$, $B^p_{\delta_0}$ and $[0,\delta_0)$ respectively, as follows: $T = \frac{d}{dt}$ in $(0,\delta_0)$ and $T_0 = 0$; Y is a lifting of T through ρ and X is a lifting of Y through f. The existence of X, Y is given by [11, Theorem 3.2]. Moreover, since T is globally integrable, then Y, X are also globally integrable, by [11, Lemma 4.8].

Let $0 < \delta_1 < \delta_0$. We define the mapping

$$r : D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \to V_{\epsilon_0},$$

such that $r(x)$ is the point of V_{ϵ_0} where the integral curve of X passing through x meets V_{ϵ_0}. We consider the link diagram:

$$V_{\epsilon_0} \xleftarrow{r} N_{\epsilon_0,\delta_1} \xrightarrow{f} S^{p-1}_{\delta_1}$$

We define

$$\Phi : D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \to (N_{\epsilon_0,\delta_1} \times [0,\delta_1]) \cup_r V_{\epsilon_0},$$

as follows:

$$\Phi(x) = \begin{cases} [\phi(x), \|f(x)\|^2], & \text{if } x \notin V_{\epsilon_0}, \\ [r(x), 0], & \text{if } x \in V_{\epsilon_0}, \end{cases}$$

being $\phi(x)$ the point of N_{ϵ_0,δ_1} where the integral curve of X passing through x meets N_{ϵ_0,δ_1}. Analogously, we also define $\Psi : D^p_{\delta_1} \to c(S^{p-1}_{\delta_1})$, as

$$\Psi(y) = \begin{cases} [\psi(y), \|y\|^2], & \text{if } y \neq 0, \\ [y_0, 0], & \text{if } y = 0, \end{cases}$$

being $\psi(y)$ the point of $S^{p-1}_{\delta_1}$ where the integral curve of Y passing through y meets $S^{p-1}_{\delta_1}$ and $y_0 \in S^{p-1}_{\delta_1}$. It is not difficult to see that Φ, Ψ are homeomorphisms which make commutative the following diagram

$$\begin{array}{ccc} D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) & \xrightarrow{f} & D^p_{\delta_1} \\ {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\Psi}\big\downarrow \\ (N_{\epsilon_0,\delta_1} \times [0,\delta_1]) \cup_r V_{\epsilon_0} & \xrightarrow{C(f,r)} & c(S^{p-1}_{\delta_1}). \end{array}$$

This proves that f is C^0-$\mathscr{A}$-equivalent to the generalized cone of the link diagram. The construction for other values of ϵ, δ can be done by using Theorem 7.1.

In the case that f has no DST, then the theorem is still valid, but we use the canonical Thom stratification of f instead of the stratification by stable types (see [11, page 32]).

Definition 7.7. Let $f : U \to V$ a good representative of a polynomial map germ $f \in \mathscr{E}(n,p)$ with II and $f^{-1}(0) \neq \{0\}$. The *link diagram* of f is the link diagram

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^{p-1}_\delta$$

given in Theorem 7.6 for $0 < \delta \ll \epsilon \ll 1$. Then, f is C^0-$\mathscr{A}$-equivalent to the generalized cone of its link diagram.

Corollary 7.8. *Let $f, g \in \mathscr{E}(n,p)$ be two FD germs with non isolated zeros. If their link diagrams are C^0-$\mathscr{A}$-equivalent, then f, g are C^0-$\mathscr{A}$-equivalent.*

Exemple 7.9. Consider a FD function germ $f \in \mathscr{E}(2,1)$ with $f^{-1}(0) \neq \{0\}$. The FD condition implies that f has isolated critical point in the origin. We fix $0 < \delta \ll \epsilon \ll 1$ as in Theorems 7.1 and 7.6. We can assume f is polynomial, hence $f^{-1}(0)$ is the algebraic curve given by $f(x,y) = 0$. Then, $V_\epsilon = f^{-1}(0) \cap D^2_\epsilon$ is made of a finite an even number $2r$ of half-branches which intersect transversally the boundary S^1_ϵ and separate the disk D^2_ϵ into $2r$ sectors, so that the sign of f alternates on consecutive sectors.

The manifold $N_{\epsilon,\delta}$ is given by the level curves $f(x,y) = \pm\delta$ in D^2_ϵ. It has $2r$ connected components, one in each sector of $D^2_\epsilon \setminus f^{-1}(0)$ and diffeomorphic to a closed interval. Moreover, he retraction map $r : N_{\epsilon,\delta} \to V_\epsilon$, when restricted to each connected component, is a diffeomorphism onto the two half-branches which bound the sector containing the connected component.

Thus, the C^0-$\mathscr{A}$-class only depends on the number of half-branches $2r$. We deduce that two functions f, g are C^0-$\mathscr{A}$-equivalent if and only if the curves $f^{-1}(0)$ and $g^{-1}(0)$ have the same number of half-branches.

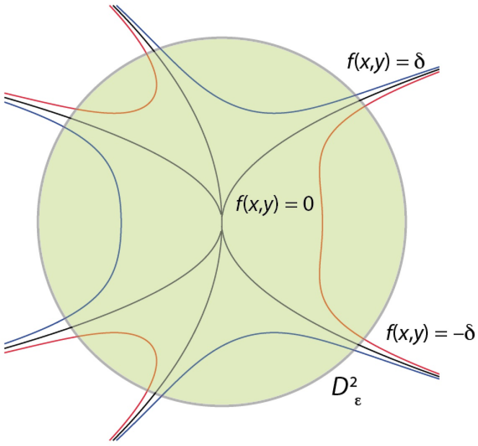


FIGURE 24. The link of a FD function $f \in \mathscr{E}(2,1)$

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